Nonlinear Optimization: Discrete optimization

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Motivations

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \mathcal{X} \;, \\ & g_j(x) \leq 0 \;, \quad j = 1, \ldots, r \;, \end{aligned}$$

where $\mathcal{X}$ is a *finite* set (e.g., $0-1$-valued vectors).

- Many problems involve integer constraints
- Applications in scheduling, resource allocation, engineering design...
- Diverse methodology for their solution, but an important subset of this methodology relies on the solution of *continuous optimization subproblems*, as well as on *duality*.

Outline

- Network optimization and unimodularity
- Examples of nonunimodular problems
- Branch-and-bound
- Lagrange relaxation

Network optimization and unimodularity

Network optimization

- Let a directed graph with set of nodes $\mathcal{N}$ and set of arcs $(i,j) \in \mathcal{A}$.
- An integer-constrained network optimization problem is:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in\mathcal{A}} a_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{\{j|(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j|(j,i)\in\mathcal{A}\}} x_{ji} = s_i \ , \forall i \in \mathcal{N} \\
& b_{ij} \leq x_{ij} \leq c_{ij} \ , \forall (i,j) \in \mathcal{A} \\
& x_{ij} \in \mathbb{N} \ .
\end{aligned}
$$

Example: transportation optimization

- Nodes are *locations* (cities, warehouses, or factories) where a certain product is produced or consumed
- Arcs are *transportation links* between the locations
- $a_{i,j}$ is the *transportation cost* per unit transported between locations i and j.
- The problem is to move the product from the production points to the consumption points at *minimum costs* while observing the capacity constraints of the transportation links
- s_i is the *supply* provided by node i to the outside world. It is equal to the difference between the total flows coming in and out.

Example: shortest path

- Given a starting node s and a destination node t, let the "supply":

$$s_i = \begin{cases} 1 & \text{if } i = s, \\ -1 & \text{if } i = t, \\ 0 & \text{otherwise.} \end{cases}$$

  and let the constraint $x_{ij} \in \{0, 1\}$.
- Let a_{ij} be the distance between locations i and j.
- Any feasible solution corresponds to a *path* between s and t
- This problem is therefore that of finding the *shortest path* between s and t.

Relaxing constraints

- The most important property of the network optimization problem is that *the integer constraint can be neglected*
- The *relaxed problem* (a LP without integer constraint) has the same optimal value as the integer-constrained original
- Great significance since the relaxed problem can be solved using efficient linear (not integer) programming algorithms.

Unimodularity property

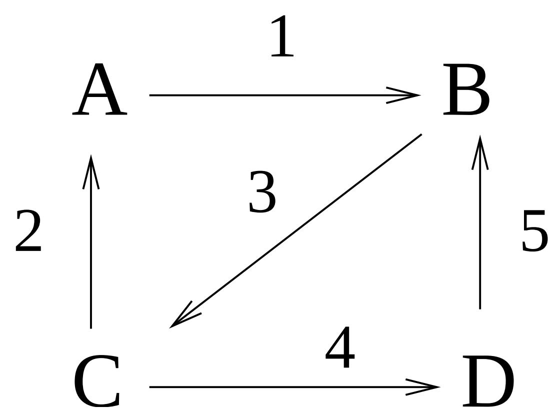
- A square matrix A with integer components is *unimodular* if its determinant is $0, 1$ or -1.
- If A is invertible and unimodular, the inverse matrix A^{-1} has integer components. Hence the solution x of the system $Ax = b$ is integer for every integer vector b.
- A rectangular matrix with integer components is called *totally unimodular* if each of its square submatrices is unimodular
- Key fact: A polyhedron $\{x|Ex = d, b \leq x \leq c\}$ has integer extreme points if E is totally unimodular and b, c and d have integer components

Unimodularity of network optimization

$$\begin{aligned} \text{minimize} \quad & a^\top x \\ \text{subject to} \quad & Ex = d\,, \quad d \leq x \leq c\,. \end{aligned}$$

- The fundamental theorem of linear programming states that the solution to a linear program is an extreme point of the polyhedron of feasible points.
- The constraint matrix for the network optimization problem is the *arc incidence matrix* for the underlying graph. We can show that it is totally unimodular (by induction, left as exercise)
- Therefore the problem is unimodular: the solution of the LP has integer values!
- However, unimodularity is an exceptional property...

Example: shortest path as a LP



$$
\begin{aligned}
\text{minimize} \quad & x_1 + x_2 + x_3 + x_4 + x_5 \\
\text{subject to} \quad & x_1 - x_2 = 1 \,, \\
& x_3 - x_1 - x_5 = 0 \,, \\
& x_2 + x_4 - x_3 = 0 \,, \\
& x_5 - x_4 = -1 \,, \\
& 0 \leq x_i \leq 1 \,, i = 1, \ldots, 5 \,.
\end{aligned}
$$

See script `shortestpath.m`

Examples of nonunimodular problems

Generalized assignment problem

- m jobs must be assigned to n machines
- If job i is performed at machine j it costs a_{ij} and requires t_{ij} time units.
- Each job must be performed in its entirety at a single machine
- Goal: find a minimum cost assignment of the jobs to the machines, given the total available time T_j at machine j.

Formalization

- Let $x_{ij} \in \{0,1\}$ indicate whether job i is assigned to machine j.
- Each job must be assigned to some machine: $\sum_{j=1}^{n} x_{ij} = 1$.
- Limit in the total working time of machine j: $\sum_{i=1}^{m} x_{ij} t_{ij} \leq T_j$
- Total cost is $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} a_{ij}$

Optimization problem

minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} a_{ij}$

subject to $\sum_{j=1}^{n} x_{ij} = 1 \;, \quad i = 1, \ldots, m$

$$\sum_{i=1}^{m} x_{ij} t_{ij} \leq T_j \;, \quad j = 1, \ldots, n \;,$$

$$x_{ij} \in \{0, 1\} \;, \quad i = 1, \ldots, m, \; j = 1, \ldots, n. \;.$$

Other problems

- *Facility location problem*: select a subset of locations from a given candidate set, and place in each of these locations a facility that will serve the needs of certain clients up to a given capacity bound (minimize the cost)
- *Traveling salesman problem*: find a minimum cost tour that visits each of N given cities exactly once and returns to the starting city.
- *Separable resource allocation problems*: optimally produce a given amount of product using n production units
- (see Bertsekas sec. 5.5)

Approaches to discrete programming

- Enumeration of the finite set of all feasible solutions, and comparison to obtain an optimal solution (rarely practical)
- Constraint relaxation and heuristic rounding:
  - neglect the integer constraints
  - solve the problem using linear/nonlinear programming methods
  - if a noninteger solution is obtained, round it to integer using a heuristic
  - sometimes, with favorable structure, clever problem formulation, and good heuristic, this works remarkably well.

Branch-and-bound

Motivations

- Combines the preceding two approaches (enumeration and constraint relaxation)
- It uses constraint relaxation and solution of noninteger problems to obtain certain lower bounds that are used to discard large portions of the feasible set
- In principle it can find an optimal (integer) solution, but this may require unacceptable long time
- In practice, usually it is terminated with a heuristically obtained integer solution, often derived by rounding a noninteger solution.

Principle of branch-and-bound

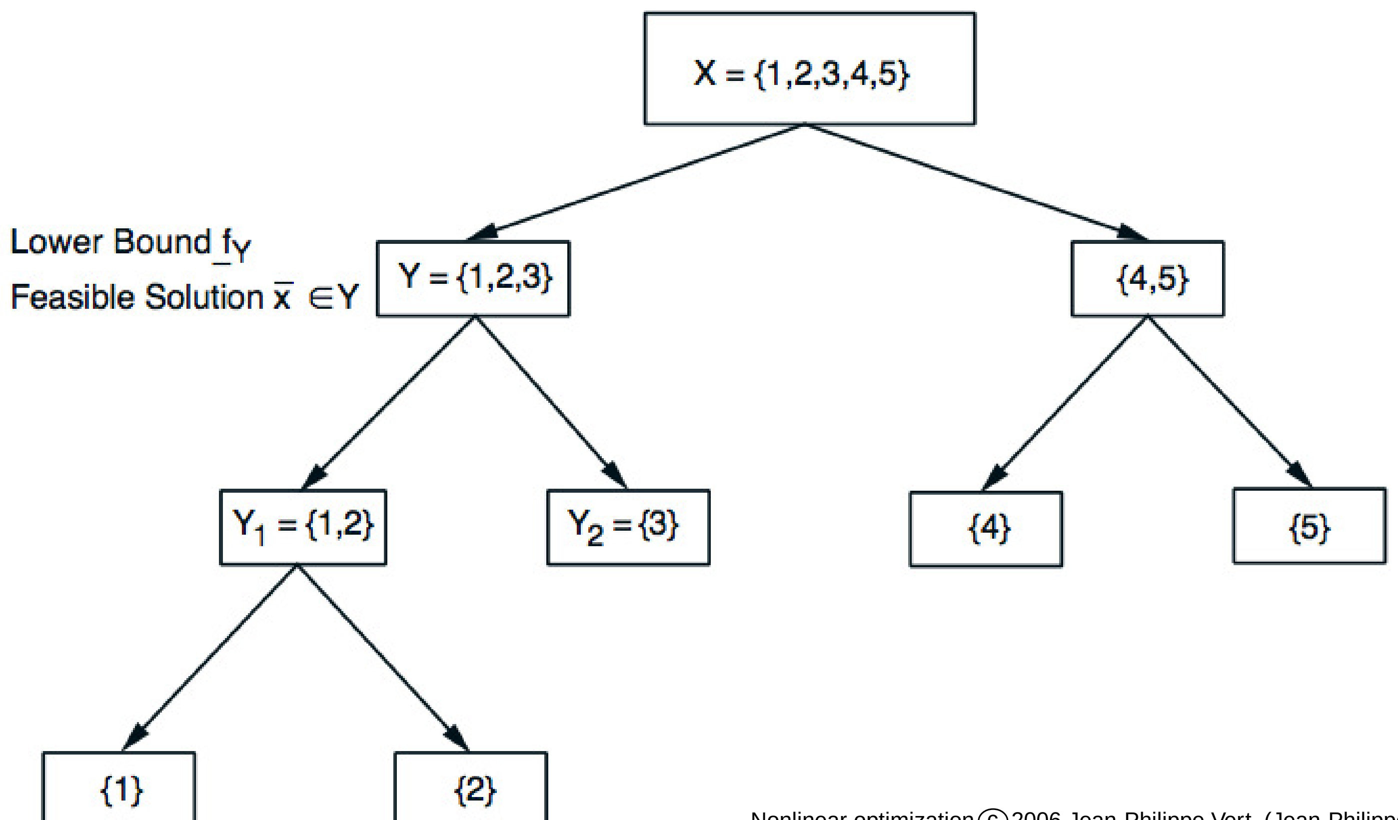
- Consider minimizing $f(x)$ over a finite set $x \in X$.
- Let Y_1 and Y_2 be two subsets of X for which we have bounds:

$$m_1 \leq \min_{x \in Y_1} f(x), \quad M_2 \geq \min_{x \in Y_2} f(x) .$$

- If $M_2 \leq m_1$ then the solutions in Y_1 may be disregarded since their cost cannot be smaller than the cost of the best solution in Y_2.

Illustration

The branch-and-bound method uses suitable upper and lower bounds, and the bounding principle to eliminate substantial portions of X. It uses a *tree*, with nodes that correspond to subsets of X, usually obtained by binary partition.



Algorithm

- Initialization: OPEN=$\{X\}$, UPPER=$+\infty$
- While OPEN is nonempty
  - Remove a node Y from OPEN
  - For each child Y_i of Y, find the lower bound m_i and a feasible solution $\bar{x} \in Y_i$.
  - If $m_i < UPPER$ place Y_i in OPEN
  - If in addition $f(\bar{x}) < UPPER$ set $UPPER = f(\bar{x})$ and mark $\bar{x}$ as the best solution found so far.
- Termination: the best solution so far is optimal.

Tight lower bounds m_i are important for quick termination!

Example: facility location

- m clients, n locations
- $x_{ij} \in \{0,1\}$ indicates that client i is assigned to location j at cost a_{ij}.
- $y_i \in \{0,1\}$ indicates that a facility is placed at location j (at cost b_j)

$$\text{minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}a_{ij} + \sum_{j=1}^{n} b_j y_j$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = 1\,, \quad i = 1,\ldots,m \\
& \sum_{i=1}^{m} x_{ij}t_{ij} \leq T_j y_j\,, \quad j = 1,\ldots,n\,, \\
& x_{ij} \in \{0,1\}\,, \quad i = 1,\ldots,m,\ j = 1,\ldots,n.\,, \\
& y_j \in \{0,1\}\,, \quad j = 1,\ldots,n.\,.
\end{aligned}$$

B&B for facility location

- It is convenient to select subsets of the form:

  $$X(J_0, J_1) = \{(x, y) \text{ feasible} : y_j = 0, \forall j \in J_0, y_j = 1, \forall j \in J_1\}$$

  where J_0 and J_1 are disjoint subsets of facility locations (i.e., for all solutions in $X(J_0, J_1)$, a facility is placed at locations in J_1, no facility is placed at the locations in J_0, and a facility may or may not be placed at the remaining locations).

- For each subset $X(J_0, J_1)$ we can obtain a lower bound and a feasible solution by solving the linear program where all integer constraints are relaxed except that the variables $y_j, j \in J_0 \cup J_1$ are fixed at either 0 or 1.

Lagrangian relaxation

Motivations

- We have seen that obtaining lower bounds on the optimal value of a discrete optimization problem is important for branch-and-bound
- Relaxing the discrete (integer) constraint is one approach to obtain such lower bounds, by transforming the integer problem into a LP or other continuous problem
- Here we consider another important method called *Lagrange relaxation*, based on weak duality.

Lagrangian relaxation

- Remember that the dual of any problem (in particular the subproblem of a node of the branch-and-bound tree) is always concave, and its maximum provides a lower bound on the optimal solution of the problem by *weak duality*
- In Lagrange relaxation, we use the dual optimal as a lower bound to the primal subproblem
- Essential for applying Lagrangian relaxation is that the dual problem is easy to solve (e.g., LP).

Comparison

Consider the problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & Ax \leq b \ , \\ & x \in X \ , \end{aligned}$$

where f is convex and X is a discrete subset of $\mathbb{R}^n$. Let f^* be the optimal primal cost.
Which bound is the tightest between constraint and Lagrange relaxation?

Comparison (cont.)

- The lower bound provided by Lagrangian relaxation is:

$$q^* = \sup_{\mu \geq 0} \inf_{x \in X} L(x, \mu) \ ,$$

  where L is the Lagrangian

- The lower bound provided by constraint relaxation is:

$$\hat{f} = \inf_{Ax \leq b} f(x)$$

- By strong duality of the problem with relaxed constraints (f is convex) we know that:

$$\hat{f} = \hat{g} = \sup_{\mu \geq 0} \inf_{x \in \mathbb{R}^n} L(x, \mu) \leq q^* \ .$$

Comparison (cont.)

- *The lower bound obtained by Lagrangian relaxation is no worse than the lower bound obtained by constraint relaxation*
- However computing the dual function may be complicated (due to other constraints), and the maximization of the dual may be nontrivial (in particular it is typically nondifferentiable).