Multiple change-points detection in multiple signal

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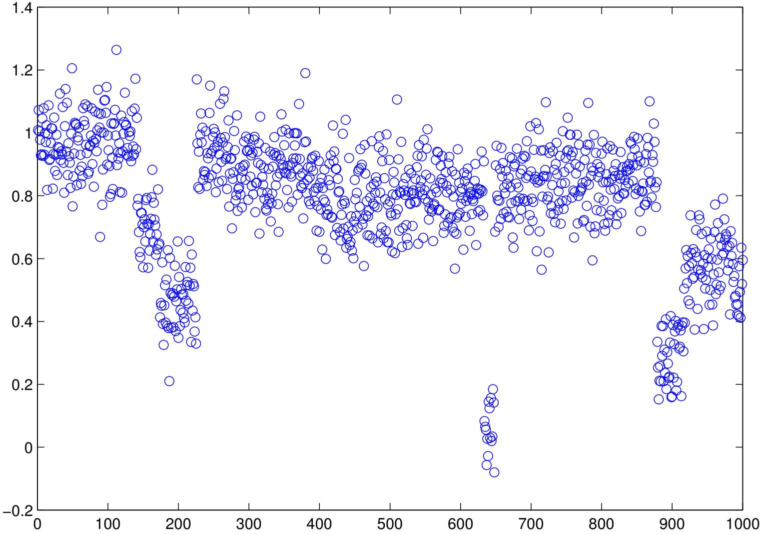
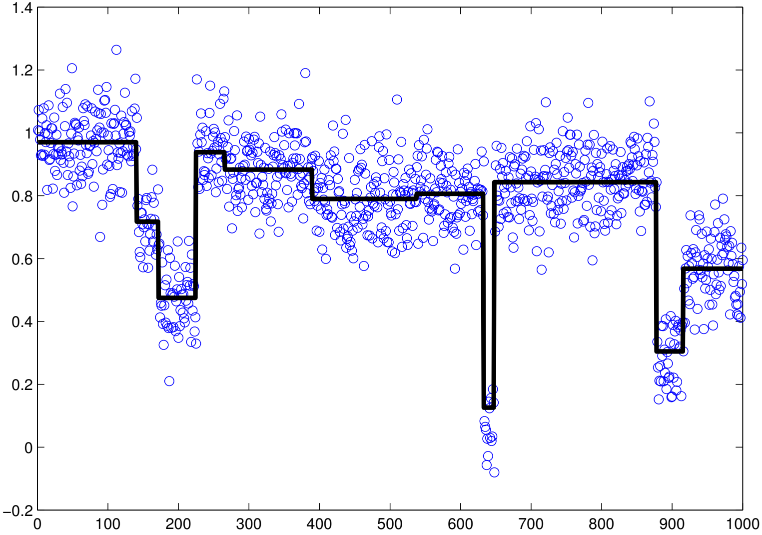
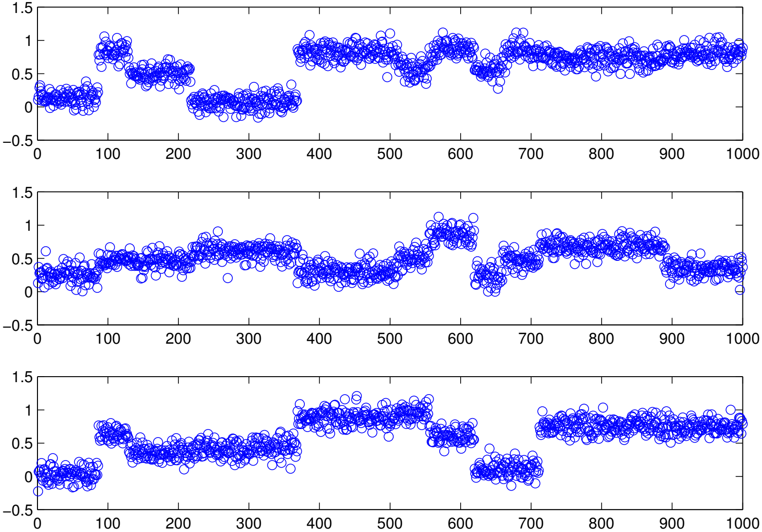
Multiple change-points detection in 1 signal



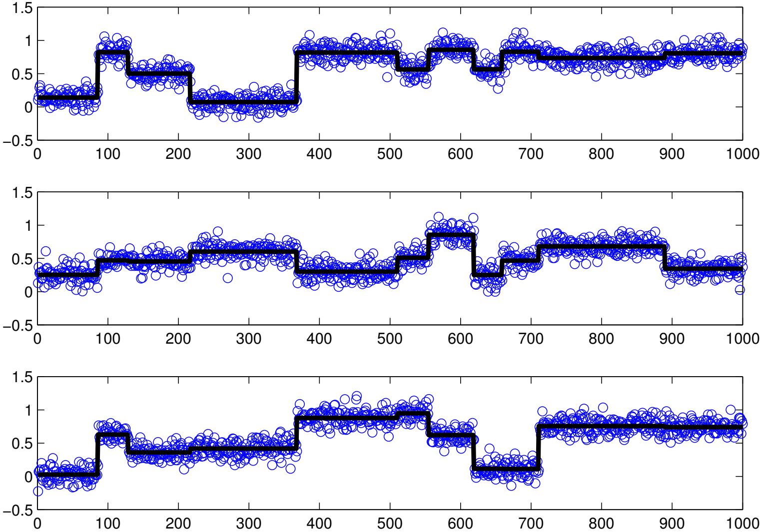
Multiple change-points detection in 1 signal



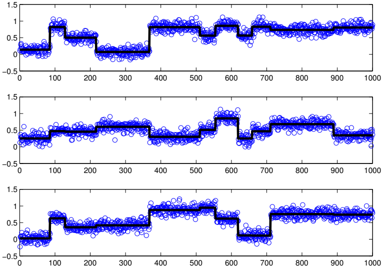
Multiple change-points detection in many signals



Multiple change-points detection in many signals

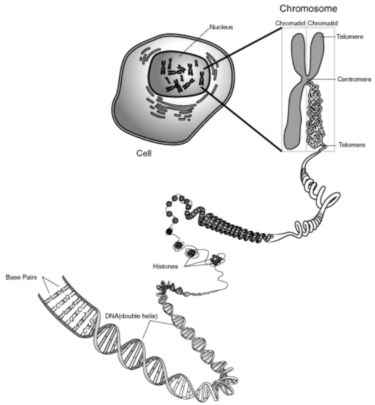


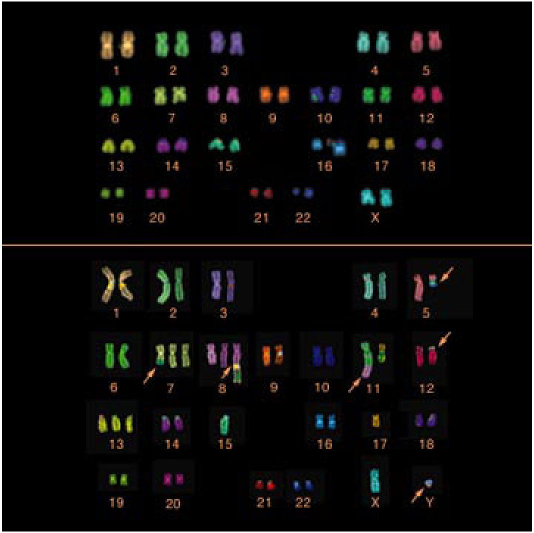
Why we care?



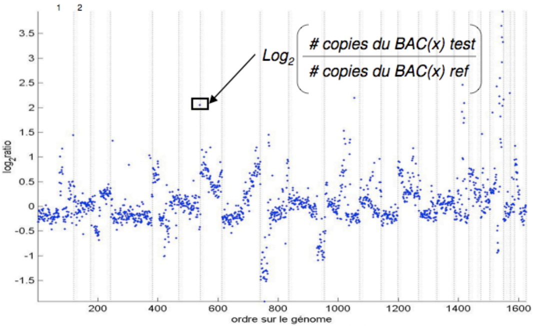
- Joint segmentation should increase the statistical power
- Applications:
  - multi-dimensional signals (multimedia, sensors...)
  - genomic profiles

Chromosomic aberrations in cancer



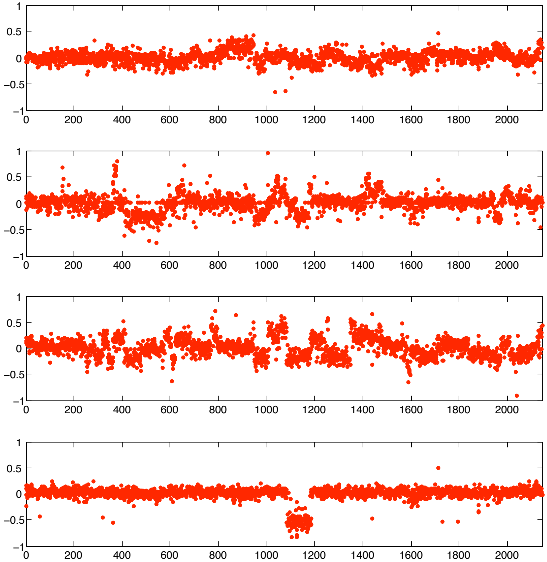


Comparative Genomic Hybridization (CGH)



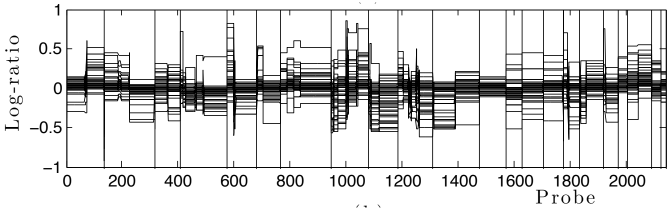
Jain et al. Genome research 2002 12:325-332

A collection of bladder tumours

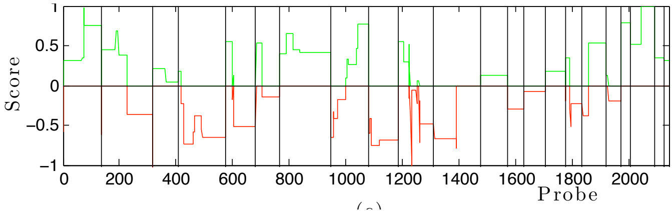


Typical applications

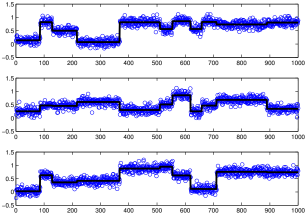
- Find frequent breakpoints in a collection of tumours (fusion genes...)
- Low-dimensional summary and visualization of the set of profiles



- Detection of frequently altered regions



What we want



1. An algorithm that scales in time and memory to
   - Profiles length: $n = 10^6 \sim 10^9$
   - Number of profiles (dimension): $p = 10^2 \sim 10^3$
   - Number of change-points: $k = 10^2 \sim 10^3$
2. A method with good statistical properties when p increases for n fixed (opposite to most existing litterature).

Segmentation by dynamic programming

- $Y \in \mathbb{R}^{n \times p}$ the signals
- Define a piecewise constant approximation $\hat{U} \in \mathbb{R}^{n \times p}$ of Y with k change-points as the solution of

$$\min_{U \in \mathbb{R}^{n \times p}} \| Y - U \|^2 \quad \text{such that} \quad \sum_{i=1}^{n-1} \mathbf{1}\left(U_{i+1,\bullet} \neq U_{i,\bullet}\right) \leq k$$

- DP finds the solution in $O(n^2 kp)$ in time and $O(n^2)$ in memory
- Does not scale to $n = 10^6 \sim 10^9$...

TV approximator for a single signal ($p = 1$)

- Replace

$$\min_{U \in \mathbb{R}^n} \| Y - U \|^2 \quad \text{such that} \quad \sum_{i=1}^{n-1} \mathbf{1}(U_{i+1} \neq U_i) \leq k$$

by

$$\min_{U \in \mathbb{R}^n} \| Y - U \|^2 \quad \text{such that} \quad \sum_{i=1}^{n-1} | U_{i+1} - U_i | \leq \mu$$

- An instance of total variation penalty (Rudin et al., 1992)
- Convex problem, fast implementations in $O(nK)$ or $O(n \log n)$ (Friedman et al., 2007; Harchaoui and Levy-Leduc, 2008; Hoefling, 2009)

TV approximator for many signals

- Replace

$$\min_{U \in \mathbb{R}^{n \times p}} \| Y - U \|^2 \quad \text{such that} \quad \sum_{i=1}^{n-1} \mathbf{1}\left(U_{i+1,\bullet} \neq U_{i,\bullet}\right) \leq k$$

by

$$\min_{U \in \mathbb{R}^{n \times p}} \| Y - U \|^2 \quad \text{such that} \quad \sum_{i=1}^{n-1} w_i \| U_{i+1,\bullet} - U_{i,\bullet} \| \leq \mu$$

Questions

- Practice: can we solve it efficiently?
- Theory: does it benefit from increasing p (for n fixed)?

TV approximator as a group Lasso problem

- Make the change of variables:

$$\gamma = U_{1,\bullet}\,,$$
$$\beta_{i,\bullet} = w_i\left(U_{i+1,\bullet} - U_{i,\bullet}\right) \quad \text{for } i = 1, \ldots, n-1\,.$$

- TV approximator is then equivalent to the following group Lasso problem (Yuan and Lin, 2006):

$$\min_{\beta \in \mathbb{R}^{(n-1)\times p}} \| \bar{Y} - \bar{X}\beta \|^2 + \lambda \sum_{i=1}^{n-1} \| \beta_{i,\bullet} \|\,,$$

where $\bar{Y}$ is the centered signal matrix and $\bar{X}$ is a particular $(n-1) \times (n-1)$ design matrix.

TV approximator implementation

$$\min_{\beta \in \mathbb{R}^{(n-1)\times p}} \| \bar{Y} - \bar{X}\beta \|^2 + \lambda \sum_{i=1}^{n-1} \| \beta_{i,\bullet} \| ,$$

Theorem

The TV approximator can be solved efficiently:

- approximately with the group LARS in $O(npk)$ in time and $O(np)$ in memory
- exactly with a block coordinate descent + active set method in $O(np)$ in memory

Proof: computational tricks...

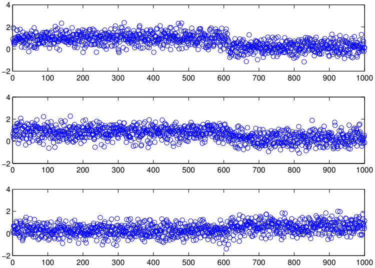
Although $\bar{X}$ is $(n-1) \times (n-1)$:

- For any $R \in \mathbb{R}^{n \times p}$, we can compute $C = \bar{X}^\top R$ in $O(np)$ operations and memory
- For any two subset of indices $A = \left(a_1, \ldots, a_{|A|}\right)$ and $B = \left(b_1, \ldots, b_{|B|}\right)$ in $[1, n-1]$, we can compute $\bar{X}^\top_{\bullet,A} \bar{X}_{\bullet,B}$ in $O\left(|A||B|\right)$ in time and memory
- For any $A = \left(a_1, \ldots, a_{|A|}\right)$, set of distinct indices with $1 \leq a_1 < \ldots < a_{|A|} \leq n-1$, and for any $|A| \times p$ matrix R, we can compute $C = \left(\bar{X}^\top_{\bullet,A} \bar{X}_{\bullet,A}\right)^{-1} R$ in $O(|A|p)$ in time and memory

Consistency for a single change-point

Suppose a single change-point:

- at position $u = \alpha n$
- with increments $(\beta_i)_{i=1,\ldots,p}$ s.t. $\bar{\beta}^2 = \lim_{k\to\infty} \frac{1}{p} \sum_{i=1}^{k} \beta_i^2$
- corrupted by i.i.d. Gaussian noise of variance σ^2



Does the TV approximator correctly estimate the first change-point as p increases?

Consistency of the unweighted TV approximator

$$\min_{U \in \mathbb{R}^{n \times p}} \| Y - U \|^2 \quad \text{such that} \quad \sum_{i=1}^{n-1} \| U_{i+1,\bullet} - U_{i,\bullet} \| \leq \mu$$

Theorem

The unweighted TV approximator finds the correct change-point with probability tending to 1 *(resp. 0) as* $p \to +\infty$ *if* $\sigma^2 < \tilde{\sigma}_\alpha^2$ *(resp.* $\sigma^2 > \tilde{\sigma}_\alpha^2$*), where*

$$\tilde{\sigma}_\alpha^2 = n\bar{\beta}^2 \frac{(1-\alpha)^2(\alpha - \frac{1}{2n})}{\alpha - \frac{1}{2} - \frac{1}{2n}} .$$

- correct estimation on $[n\epsilon, n(1-\epsilon)]$ with $\epsilon = \sqrt{\frac{\sigma^2}{2n\bar{\beta}^2}} + o(n^{-1/2})$.
- wrong estimation near the boundaries

Consistency of the weighted TV approximator

$$\min_{U \in \mathbb{R}^{n \times p}} \| Y - U \|^2 \quad \text{such that} \quad \sum_{i=1}^{n-1} w_i \| U_{i+1,\bullet} - U_{i,\bullet} \| \leq \mu$$

Theorem

The weighted TV approximator with weights

$$\forall i \in [1, n-1] \,, \quad w_i = \sqrt{\frac{i(n-i)}{n}}$$

correctly finds the first change-point with probability tending to 1 *as* $p \to +\infty$.

- we see the benefit of increasing p
- we see the benefit of adding weights to the TV penalty

Proof sketch

- The first change-point $\hat{i}$ found by TV approximator maximizes $F_i = \| \hat{c}_{i,\bullet} \|^2$, where

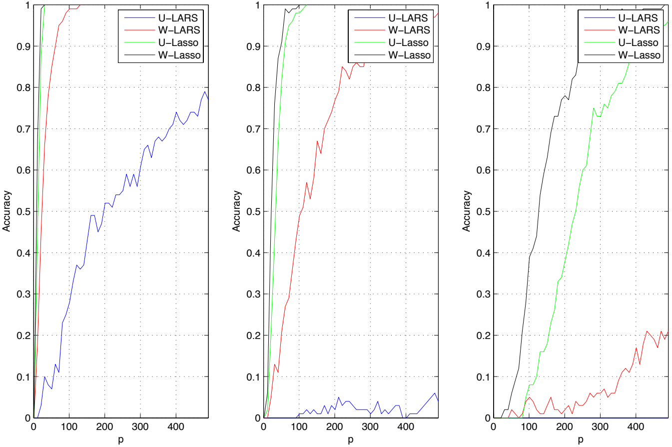
$$\hat{c} = \bar{X}^\top \bar{Y} = \bar{X}^\top \bar{X} \beta^* + \bar{X}^\top W\,.$$

- $\hat{c}$ is Gaussian, and F_i is follows a non-central χ^2 distribution with

$$G_i = \frac{EF_i}{p} = \frac{i(n-i)}{nw_i^2}\sigma^2 + \frac{\bar{\beta}^2}{w_i^2 w_u^2 n^2} \times \begin{cases} i^2\,(n-u)^2 & \text{if } i \leq u\,, \\ u^2\,(n-i)^2 & \text{otherwise.} \end{cases}$$

- We then just check when $G_u = \max_i G_i$

Consistent estimation of more change-points?



$n = 100, k = 10, \bar{\beta}^2 = 1, \sigma^2 \in \{0.05; 0.2; 1\}$

Conclusion

- A new convex formulation for multiple change-point detection in multiple signals
- Better estimation with more signals
- Importance of weights
- Efficient approximate (gLARS) and exact (gLASSO) implementations; GLASSO more expensive but more accurate
- Consistency for the first $K > 1$ change-points observed experimentally but technically tricky to prove.