Large Scale Optimization for ML

3 vignettes: GPU, Automatic Differentiation Distributed.

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Machine Learning as Optimization

Machine Learning often boils down to minimizing

variable: *parameter* which describes the machine.

objective: *fitting error* with respect to data + *regularization*

$$\boxed{\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{l}(f_{\boldsymbol{\theta}}(x_i), y_i) + \boldsymbol{\psi}(\boldsymbol{\theta})}$$

interpretation: *likelihood* + *prior* on parameter

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Computing this gradient will often cause a **BIG** problem:

$$\boxed{g = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} \nabla_{\boldsymbol{\theta}}[\boldsymbol{l}(f_{\boldsymbol{\theta}}(x_i), y_i)]}$$

Machine Learning as Optimization

Issue 1: Parameter size

Very large *parameter* vector.
for NN, this can be ~10^9.
Even one gradient is costly.

Issue 2: Model complexity

Parameters define extremely complex *functions*. How can we compute gradients?

Issue 3: Dataset size

Single machine not adequate.
Parallelism required.

Machine Learning as Optimization

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Very large *parameter* vector.
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GPUs

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Automatic Differentiation

Issue 3: Dataset size

Single machine not adequate.
Parallelism required.

Distributed Computations

Self-introduction

- **ENSAE ('01) / MVA / Phd. ENSMP / Japan & US**
  - post-doc then hedge-fund in Japan ('05~'08)
  - Lecturer @ Princeton University ('09~'10)
  - Assoc. Prof. @ Kyoto University ('10~'16)
  - Prof @ ENSAE since 9/'16.
- **Active in ML community, stats/optim flavor.**
  - Attend & publish regularly in *NIPS* & *ICML.*
- **Interests**
  - Optimal transport, kernel methods, time series.

Summary

1. Basics
    - Link between ML - Optimisation. (R)(E)RM problems
2. GPUs
3. Automatic differentiation
4. Distributed optimization

list of ingredients in ML

$$\{(x_1, y_1), \ldots, (x_n, y_n)\} \in (\mathcal{X} \times \mathcal{Y})^n$$

samples from $p \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$

loss function $l : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+$

function class $\mathcal{F} = \{f_\theta : \mathcal{X} \to \mathcal{Y}, \theta \in \Theta\}$

regularizer $\psi : \Theta \to \mathbb{R}_+$

list of ingredients in ML

$$\{(x_1, y_1), \dots, (x_n, y_n)\} \in (\mathcal{X} \times \mathcal{Y})^n$$

$n \approx \infty$

$\dim(\mathcal{X}) \approx \infty$

samples from $p \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$

loss function $l : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+$

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regularizer $\psi : \Theta \to \mathbb{R}_+$

Goal of Batch ML

1. The elusive golden standard: Risk Minimization

$$\boxed{\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \mathbb{E}_p[\boldsymbol{l}(f_{\boldsymbol{\theta}}(X), Y)]}$$

2. The naive alternative: Empirical Risk Minimization

$$\boxed{\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{l}(f_{\boldsymbol{\theta}}(x_i), y_i)}$$

Supervised ML

3. The reasonable compromise

$$\boxed{\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{l}(f_{\boldsymbol{\theta}}(x_i), y_i)}$$

From an optimization point of view:

- parameter size is huge.
- loss and regularizer functions might be ugly.
- n points might be too much for a single RAM machine (~256Gb *vs.* a few terabytes of more for modern datasets).

Supervised ML

3. The reasonable compromise

$$\boxed{\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{l}(f_{\boldsymbol{\theta}}(x_i), y_i) + \boldsymbol{\psi}(\boldsymbol{\theta})}$$

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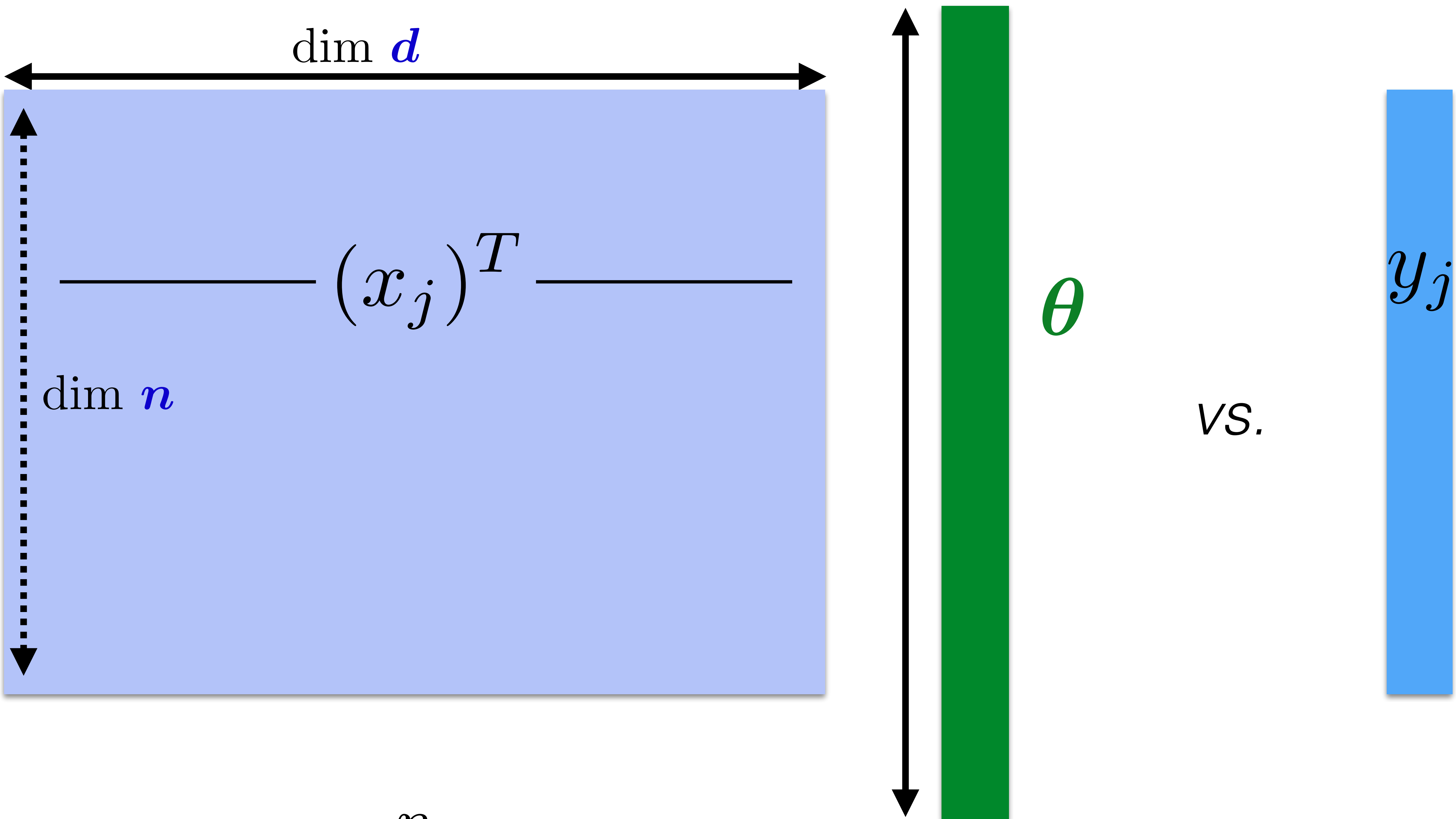
Example: Regression (Regularized)



$$\min_{\theta,b} \frac{1}{n}\sum_{j=1}^{n} (x_j^T\theta + b - y_j)^2 + \lambda \|\theta\|_q^q$$

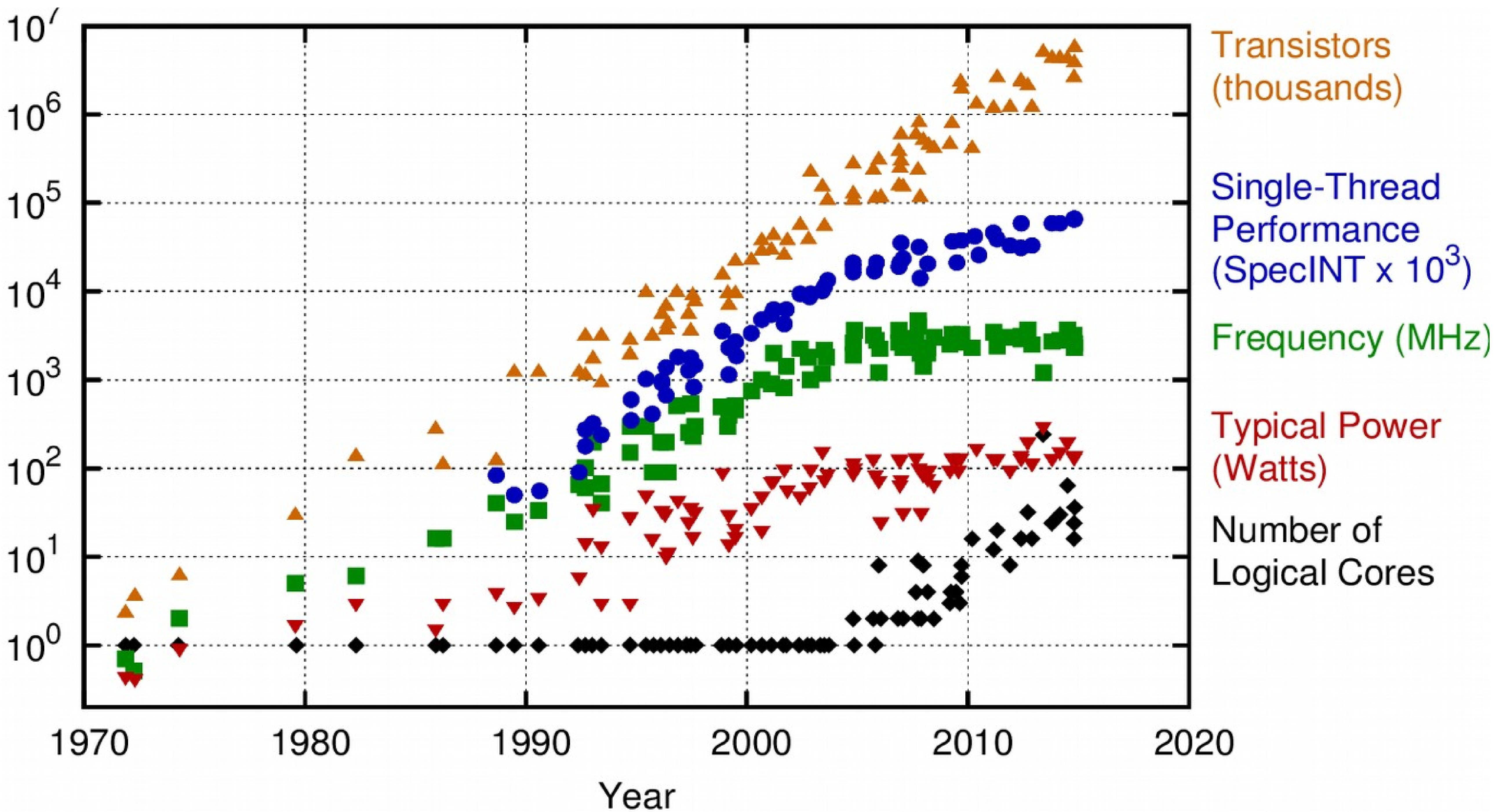
GPUs

Moore’s Law

“The complexity for minimum component costs has increased at a rate of roughly a factor of two per year. Certainly over the short term this rate can be expected to continue”
Gordon Moore (Intel), 1965

“OK, maybe a factor of two every two years.”
Gordon Moore (Intel), 1975 [paraphrased]

Moore’s Law



Original data up to the year 2010 collected and plotted by M. Horowitz, F. Labonte, O. Shacham, K. Olukotun, L. Hammond, and C. Batten
New plot and data collected for 2010-2015 by K. Rupp

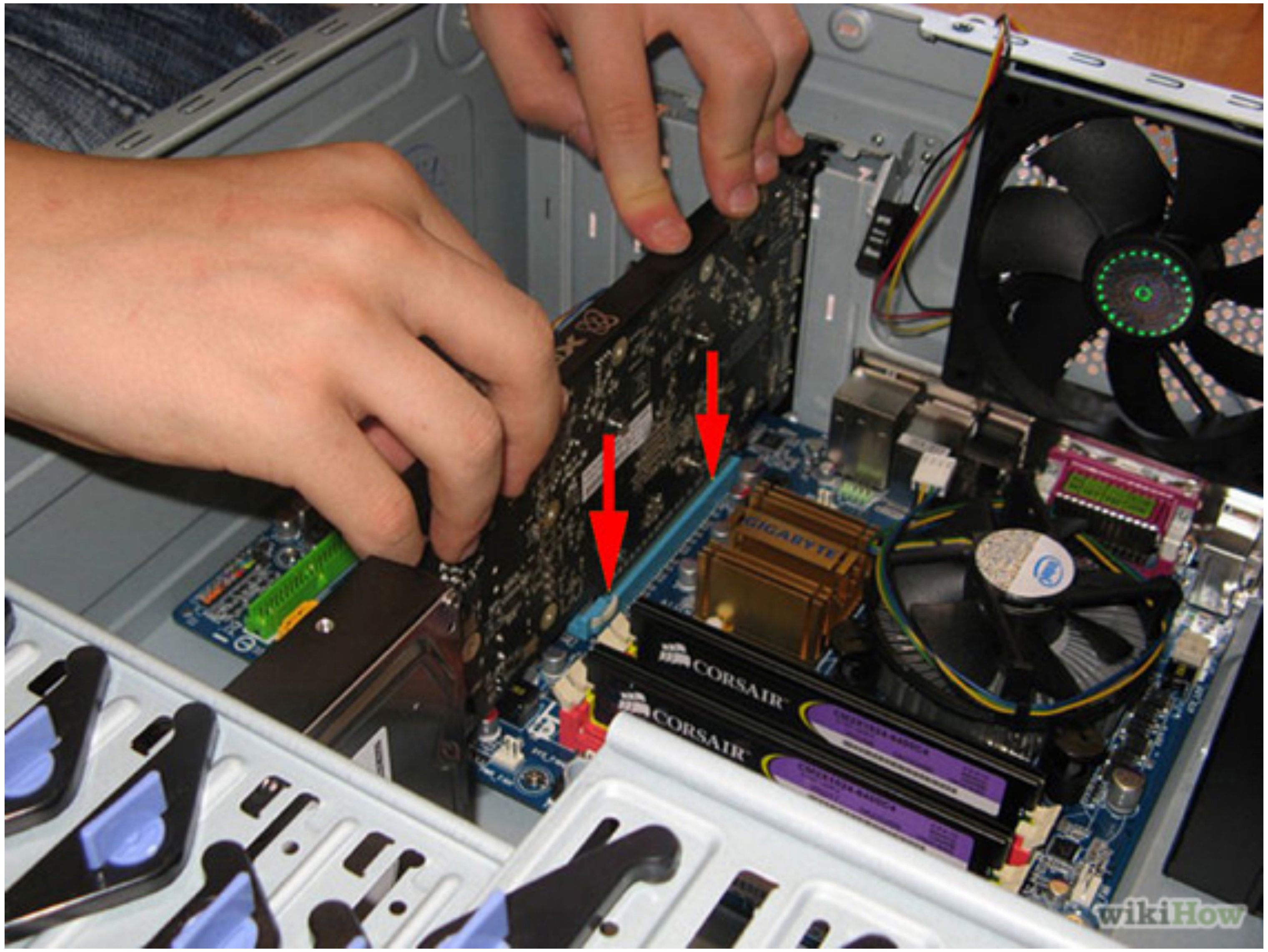
Solution: GPU

used to be a small piece of hardware…

GPU = Graphics Processing Unit

Solution: GPU

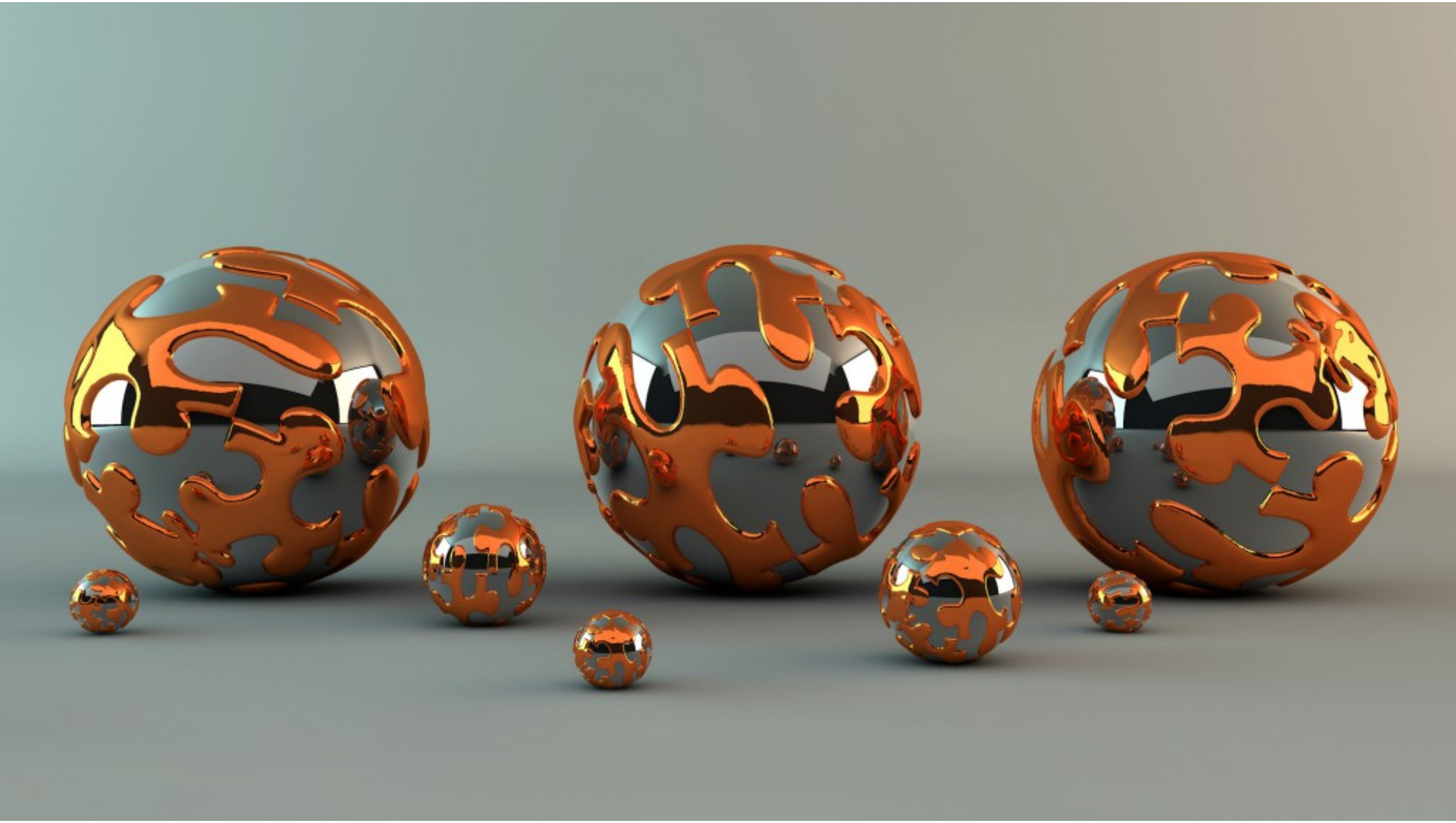
… plugged into computer, with video output…



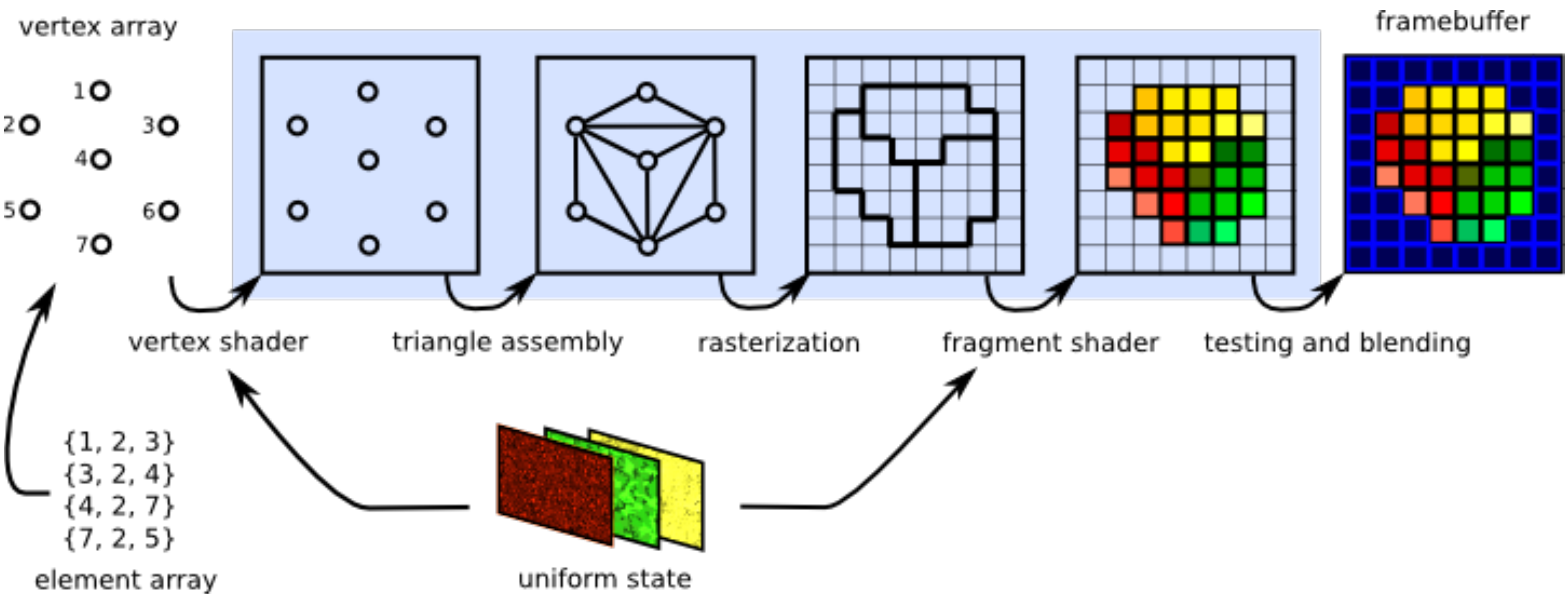
Solution: GPU

…of interest to gamers and video editors.

Graphics

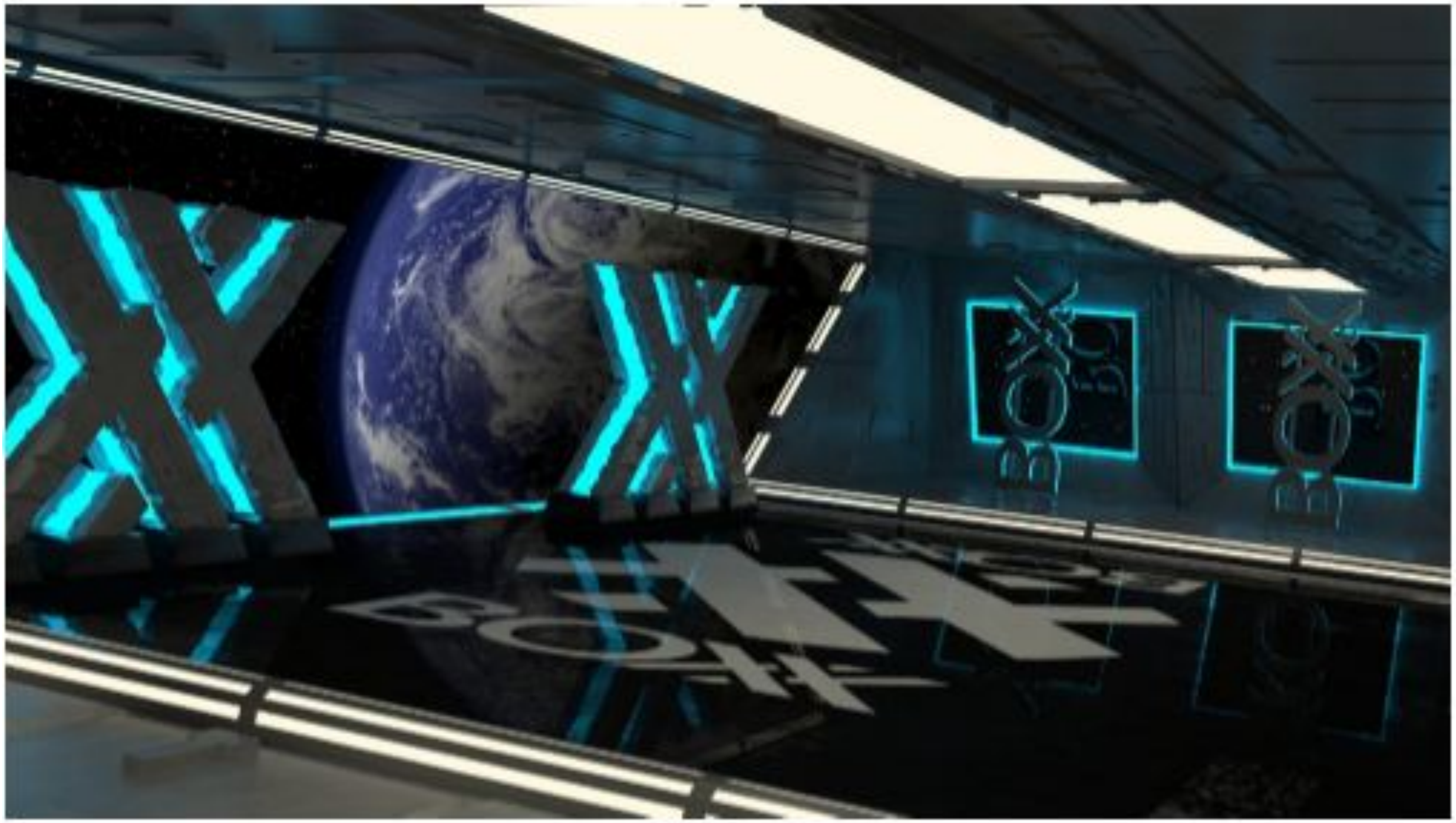


Graphics


vertex array
1
2
3
4
5
6
7
framebuffer
vertex shader
triangle assembly
rasterization
fragment shader
testing and blending
{1, 2, 3}
{3, 2, 4}
{4, 2, 7}
{7, 2, 5}
element array
uniform state

3D Rendering

Rendered with V-Ray Advanced CPU



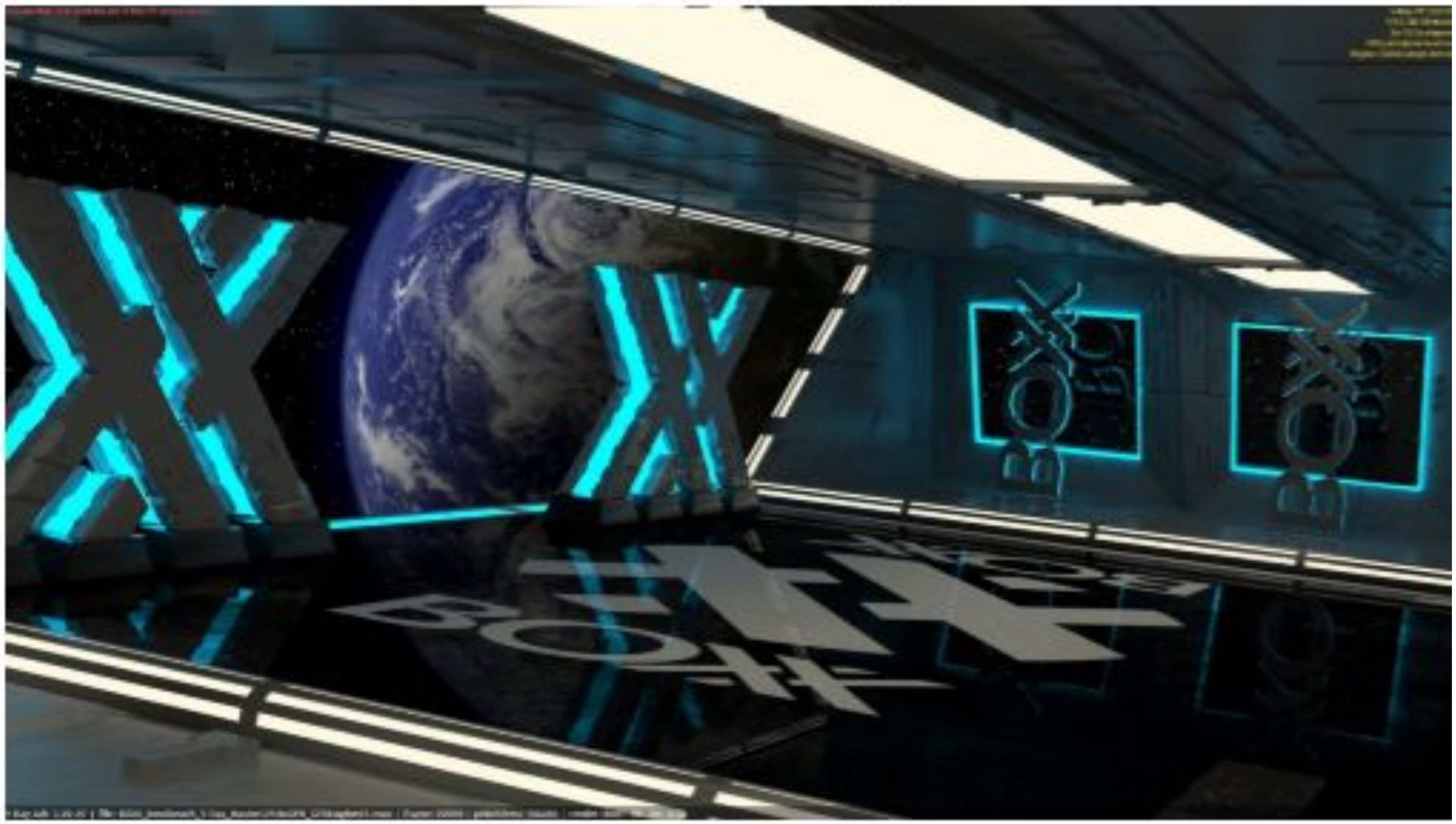
3.4 GHz 8 core Intel® Xeon®

Image Quality = 11.35

Render Time = 19 minutes 11 seconds

3D Rendering

Rendered with V-Ray RT GPU



High-end NVIDIA GPU with 2688 CUDA cores
Image Quality = 11.35
Render Time = 3 minutes 4 seconds

What are GPUs

Definition: GPU

A **programmable logic chip** (processor) specialized for **display functions.** The GPU renders images, animations and video for the computer's screen. GPUs are located on plug-in cards, in a chipset on the motherboard or in the same chip as the CPU.

A GPU performs parallel operations. Although it is used for 2D data as well as for zooming and panning the screen, a GPU is essential for smooth decoding and **rendering of 3D animations.**

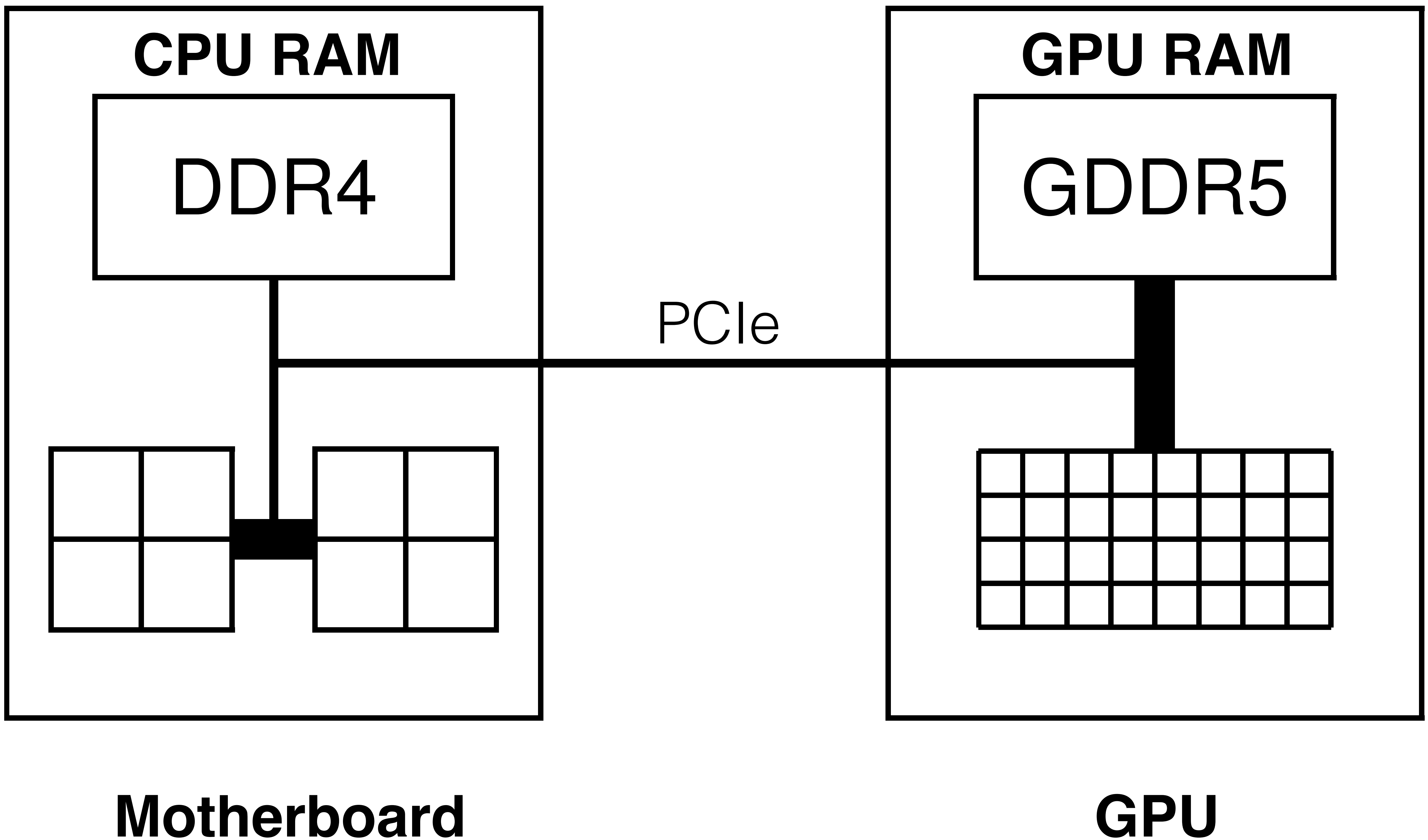
What are **GP**GPUs

Definition: **GP**GPU

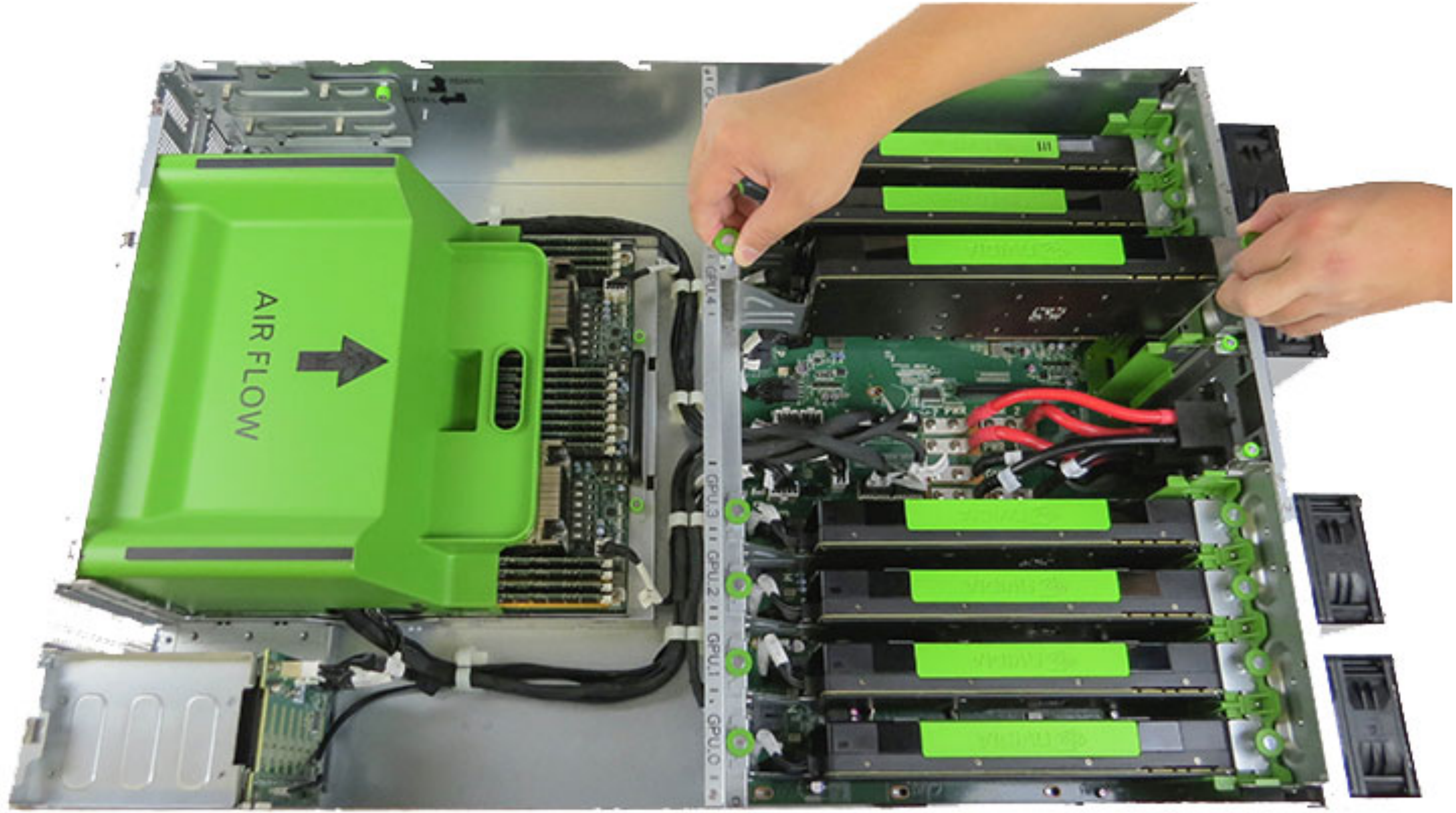
Using a GPU for general-purpose (**GP**) parallel processing applications rather than rendering images for the screen.

For fast results, applications such as sorting, **matrix algebra**, image processing and physical modeling require multiple sets of data to be processed in parallel.

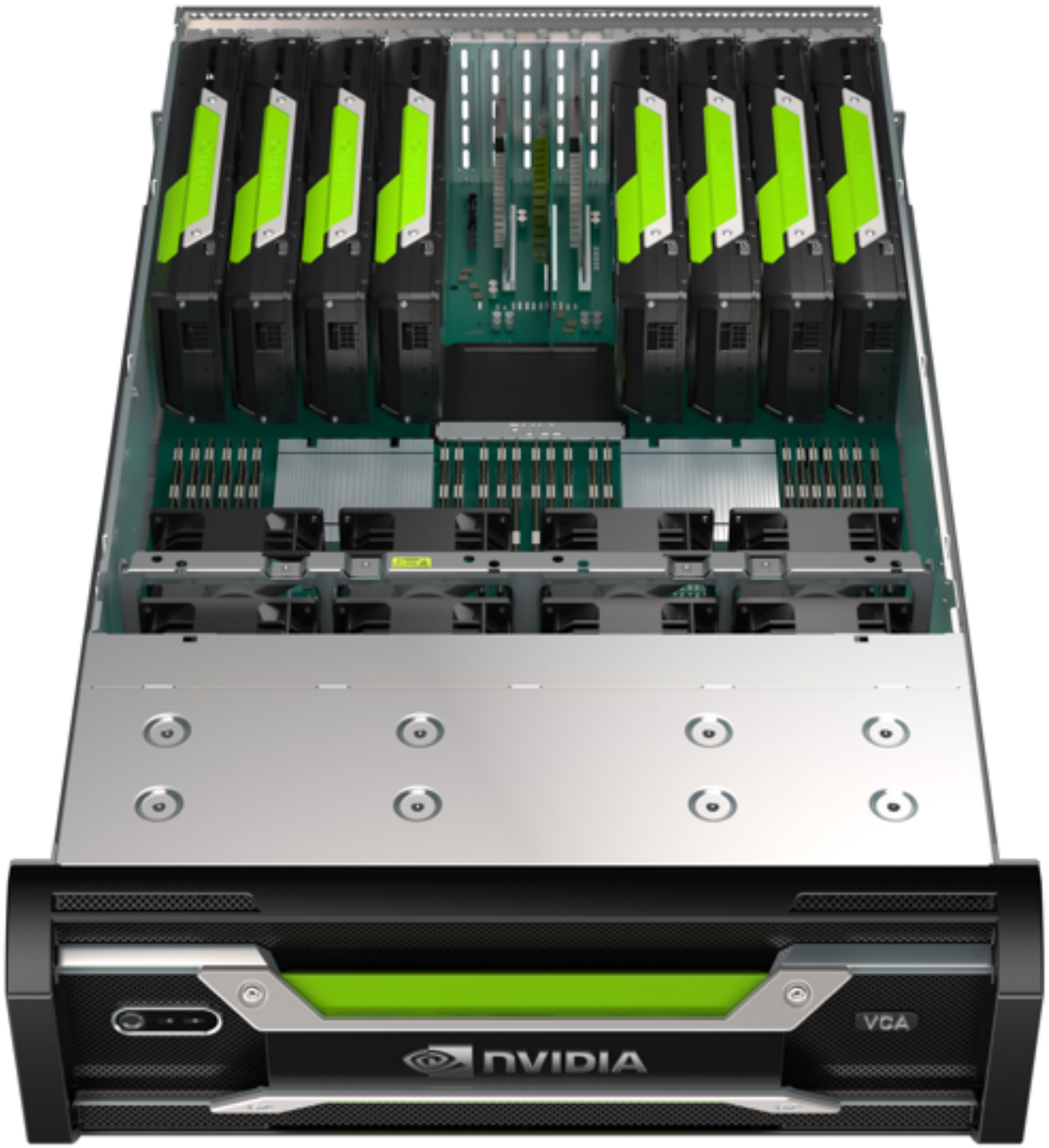
At very basic level...



In the real world



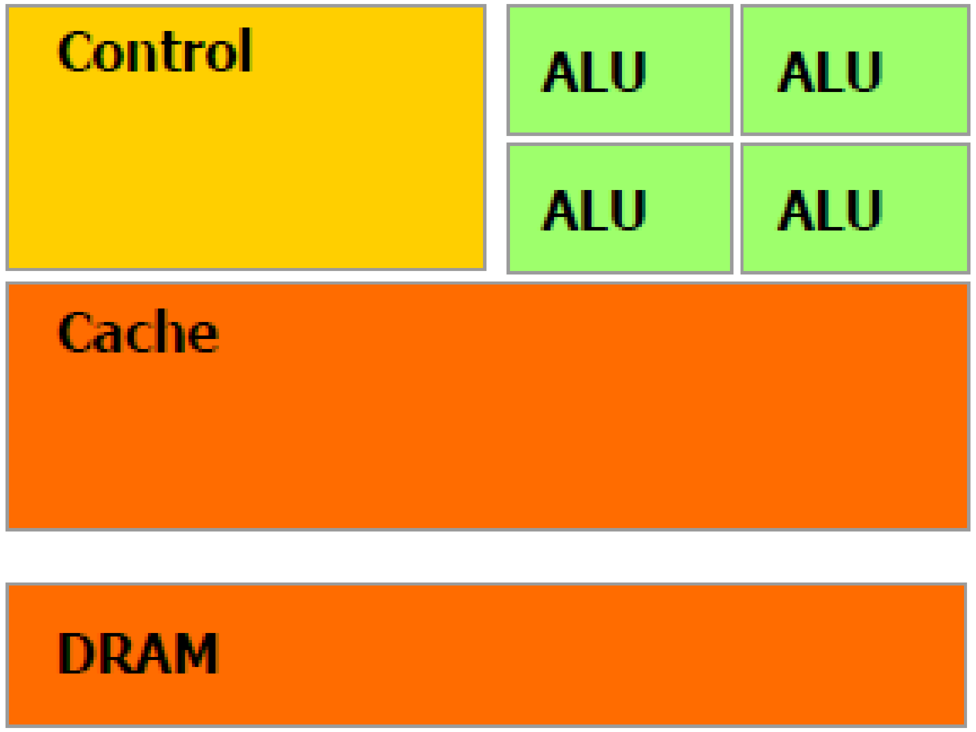
In the real world



CPU

Single Instructions, Multiple Data (SIMD)

large data-**caching**
large flow **control units**
few Arithmetic Logical Units (ALU, cores), but fast



Example: Intel Xeon E5-2670 CPU

8 cores (16 threads)
2.6 GHz
2.3 billion transistors
20 MB on chip cache
<u>Flexible</u> DRAM size

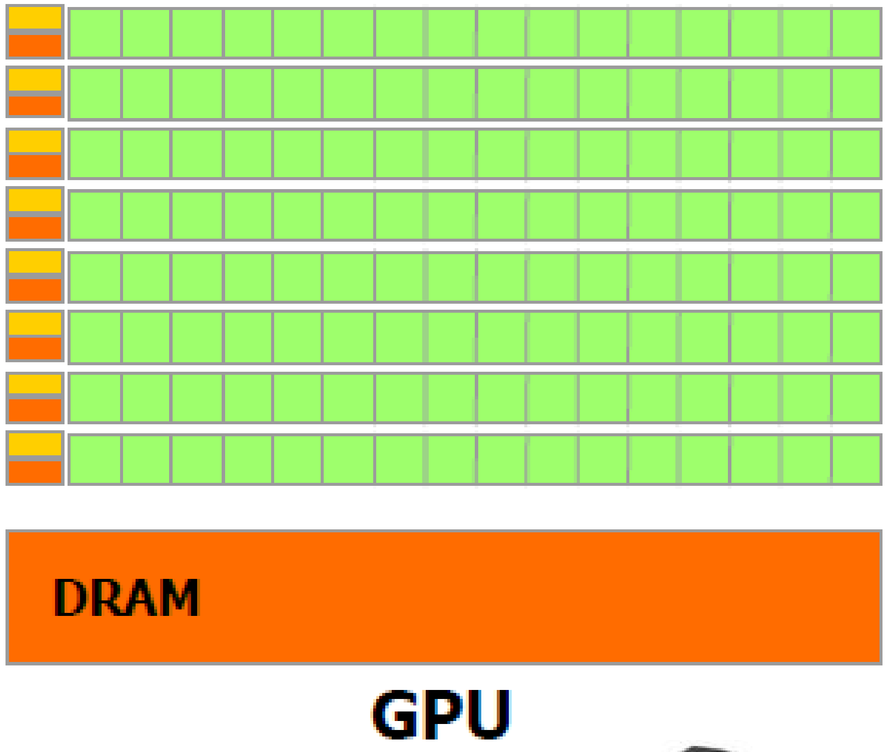
GPU

Single Instructions, Multiple Threads (SIMT)

small **cache, control flow**

Many ALUs (cores), slow.

Highly parallel.



Example: Kepler K20x GPU

2688 (14 x 192) cores

0.73 GHz

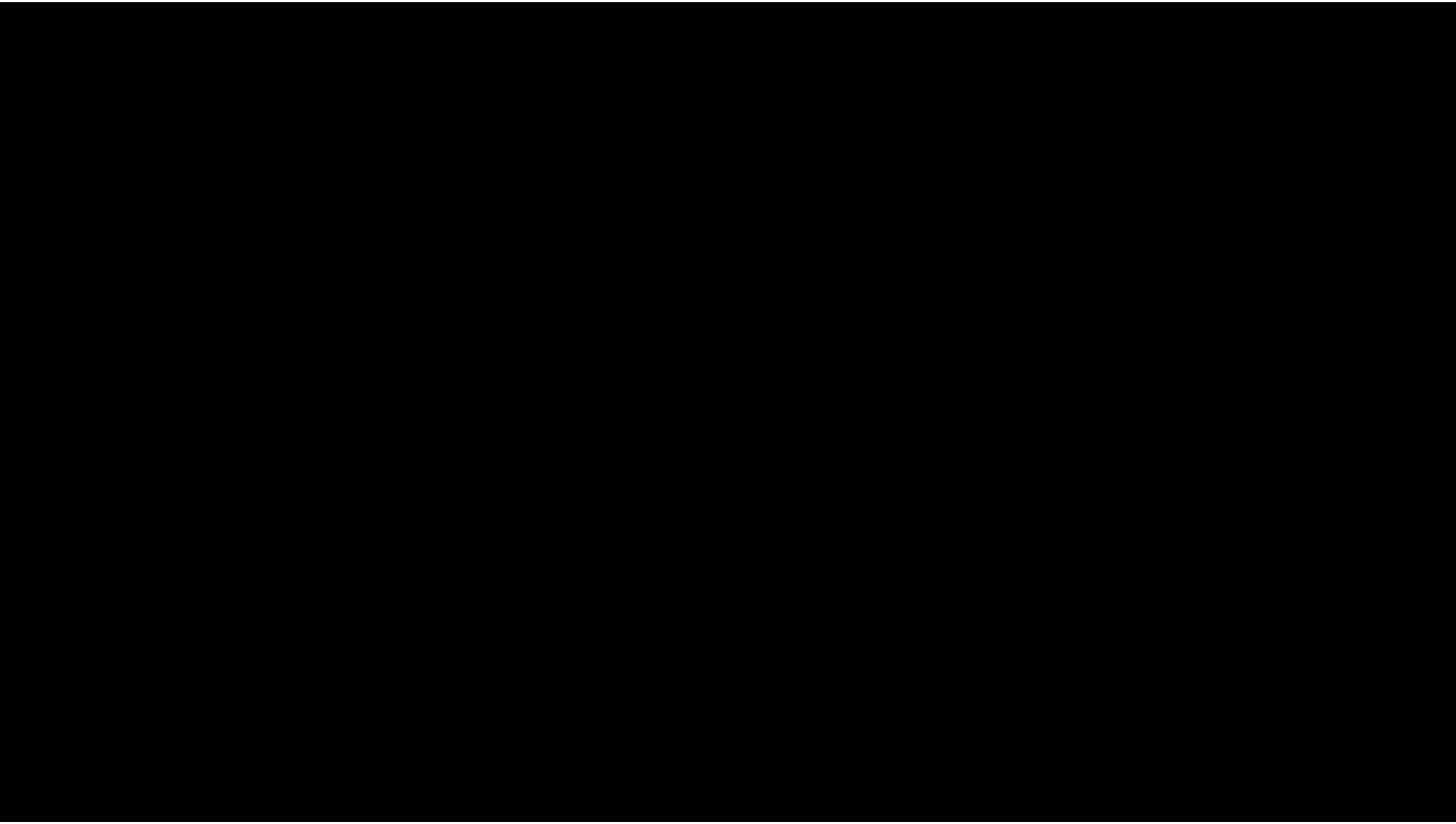
28nm features

7.1 billion transistors

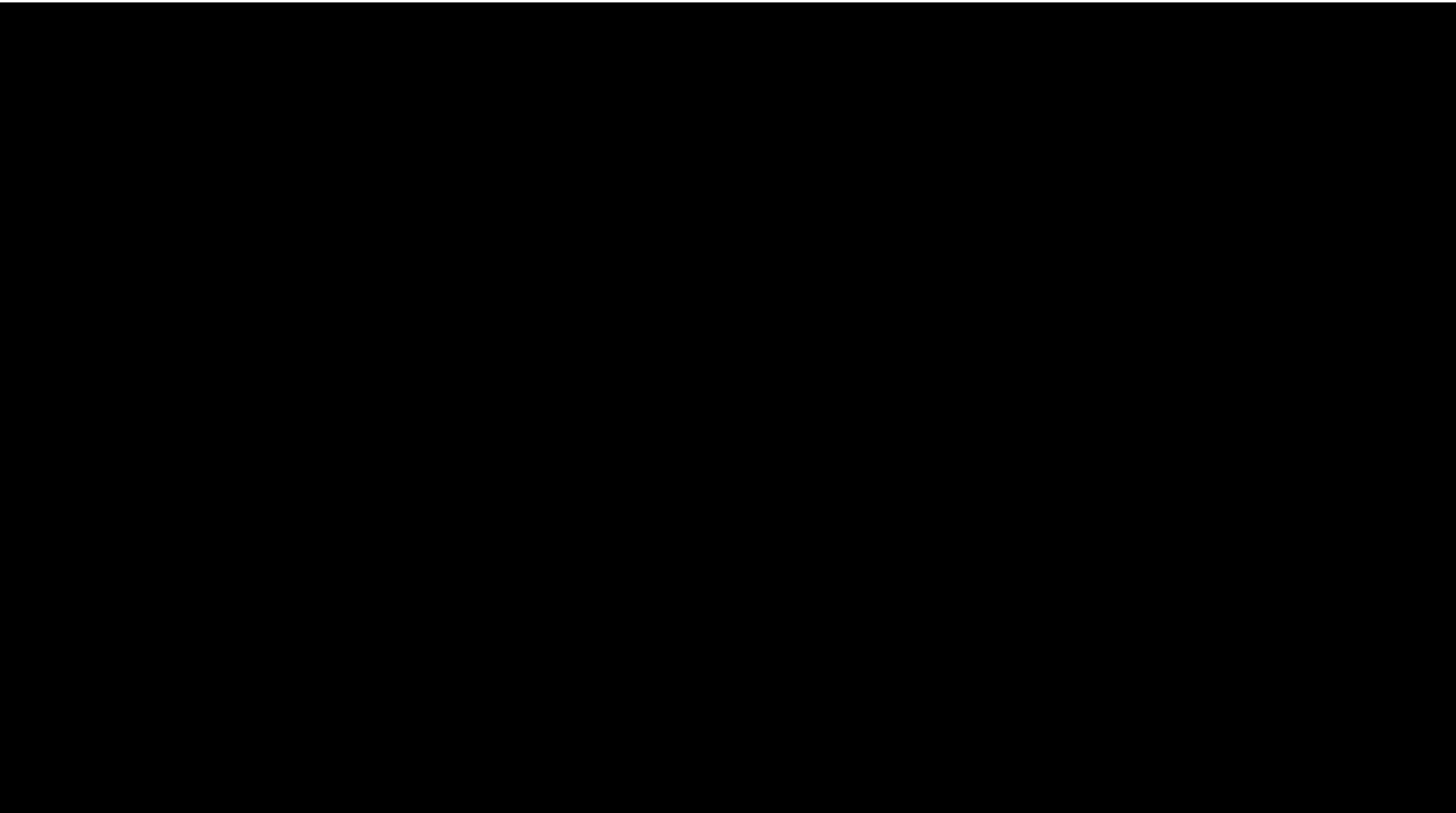
1.5 MB on-chip L2 cache

Only 6GB on chip memory

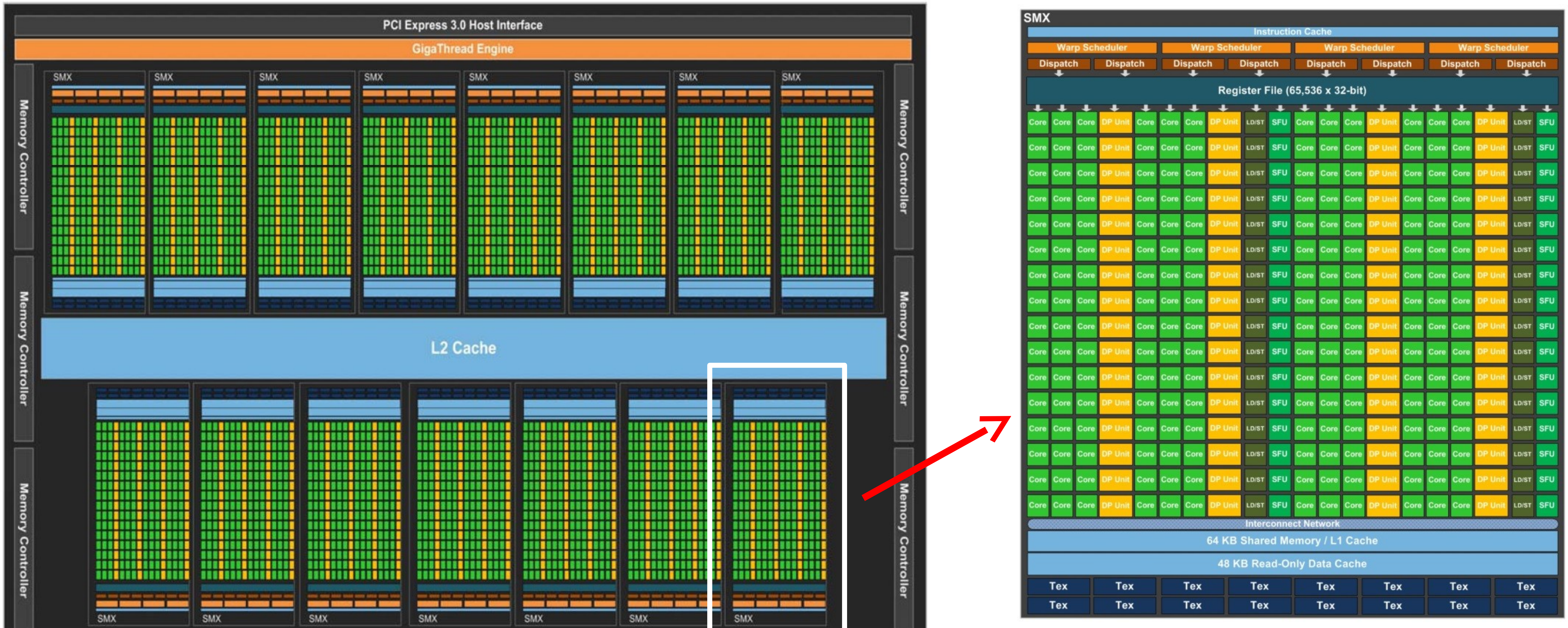
GPU *vs.* CPU



GPU *vs.* CPU



GPU Example: Kepler



SMX: 192 single-precision CUDA cores, 64 double-precision units, 32 special function units (SFU), and 32 load/store units (LD/ST).

Set of 14~15 SIMD Streaming Multiprocessors (SMX)

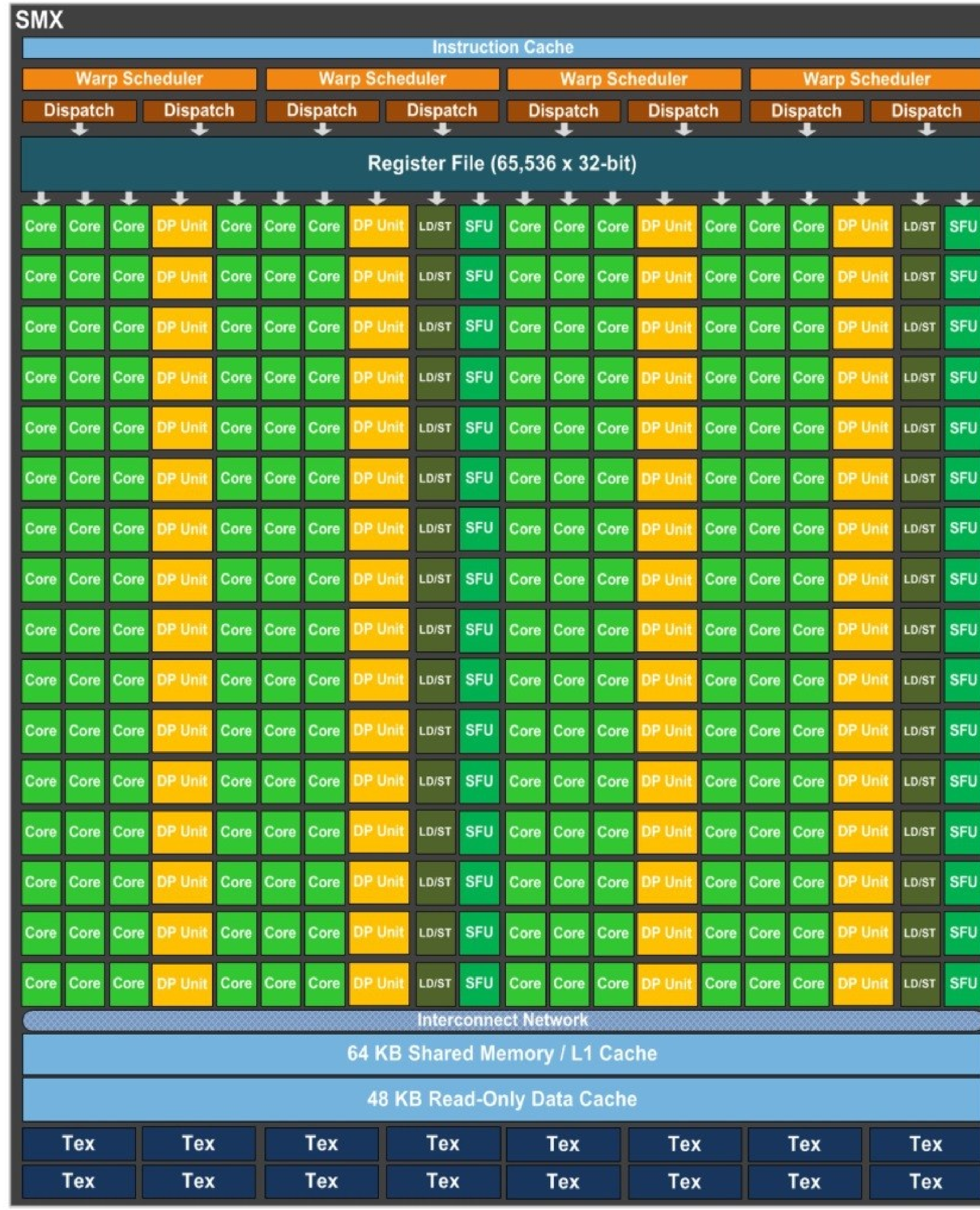
Each Multiprocessor has 192 cores, 64k L1 Cache.

Each SMX can handle up to 2000 threads.

GPU Example: Kepler

One SMX
12 x 16=192 cores
32 Special Function Units
32 Load/Store Units
64 Double Precision Units

64k shared memory



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GPU

GPU	G80	GT200	Fermi	Kepler
Transistors	681 million	1.4 billion	3.0 billion	7.0 billion
CUDA Cores	128	240	512 @ 1.15 GHz	2688 @ 0.73 GHz
Double Precision Floating Point Capability	None	30 FMA ops / clock	256 FMA ops /clock	1344 FMA ops/clock
Single Precision Floating Point Capability	128 MAD ops/clock	240 MAD ops / clock	512 FMA ops /clock	2688 FMA ops/clock
Special Function Units (SFUs) / SM	2	2	4	32
Warp schedulers (per SM)	1	1	2	2
Shared Memory (per SM)	16 KB	16 KB	Configurable 48 KB or 16 KB	Configurable 48 KB, 16 KB or 32 KB
L1 Cache (per SM)	None	None	Configurable 16 KB or 48 KB	Configurable 48 KB, 16 KB or 32 KB
L2 Cache	None	None	768 KB	1.5 MB
ECC Memory Support	No	No	Yes	Yes
Concurrent Kernels	No	No	Up to 16	Up to 32 + Dyn. Parallel
Load/Store Address Width	32-bit	32-bit	64-bit	64-bit

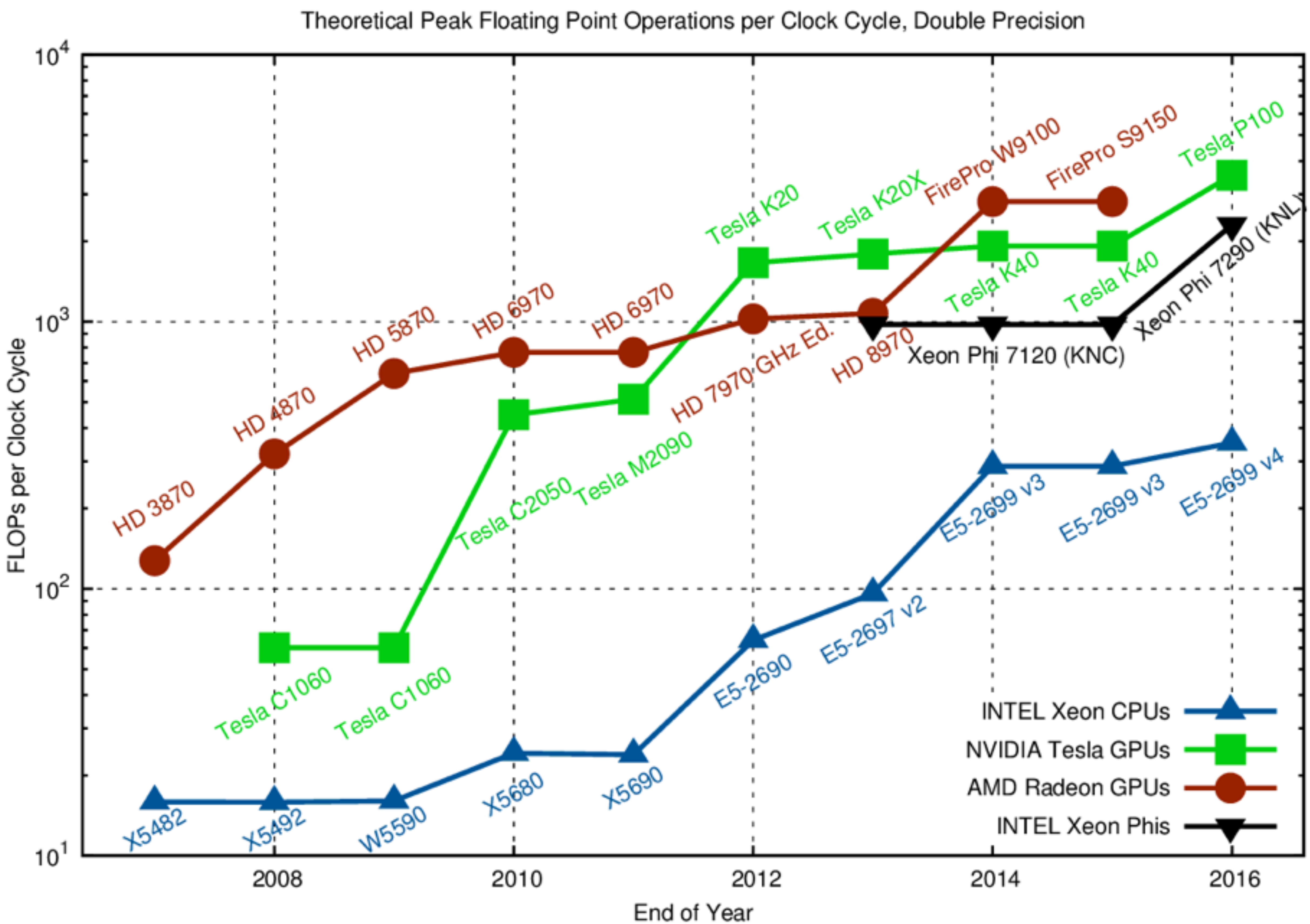
GPU

Because GPUs were designed to apply the same shading function to many pixels simultaneously, GPUs can be used to apply the same **simple** *function to many data points simultaneously*

How simple?

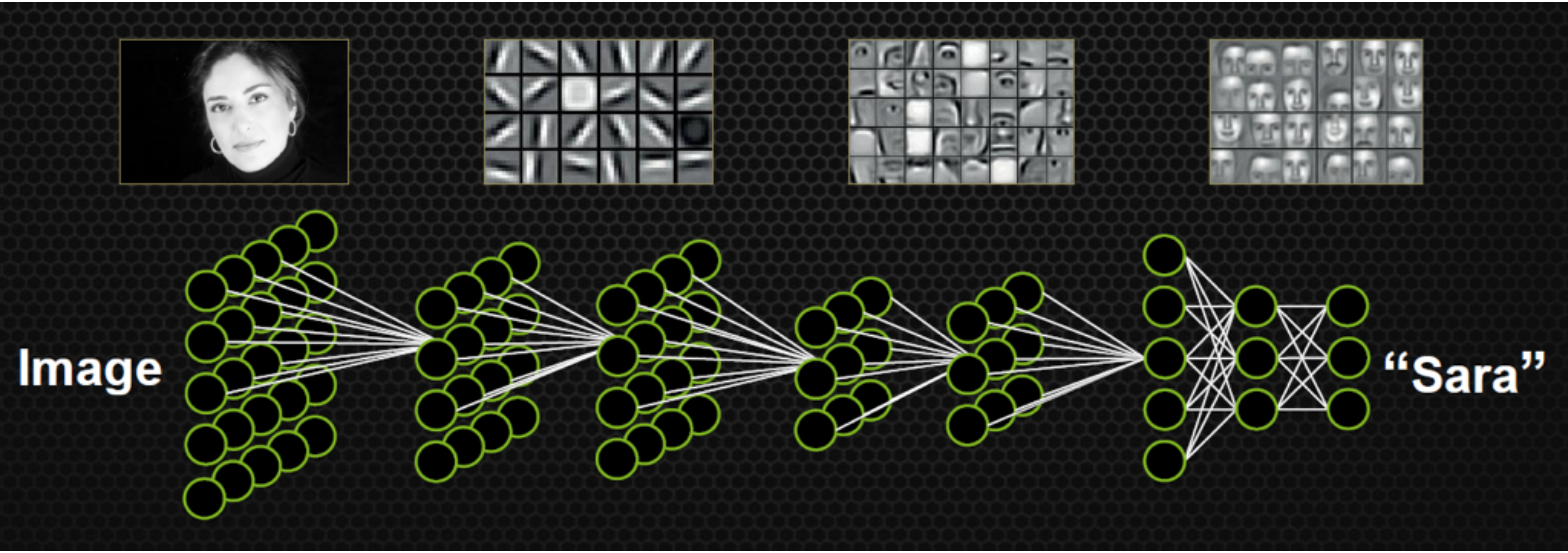
Essentially, matrix algebra and special functions on each element (exp, log, sin etc...)

How fast?



Theoretical Peak Floating Point Operations per Clock Cycle, Double Precision
FLOPs per Clock Cycle
End of Year
10^4
10^3
10^2
10^1
2008
2010
2012
2014
2016
HD 3870
HD 4870
HD 5870
HD 6970
HD 6970
HD 7970 GHz Ed.
HD 8970
FirePro W9100
FirePro S9150
Tesla C1060
Tesla C1060
Tesla C2050
Tesla M2090
Tesla K20
Tesla K20X
Tesla K40
Tesla K40
Tesla P100
Xeon Phi 7120 (KNC)
Xeon Phi 7290 (KNL)
X5482
X5492
W5590
X5680
X5690
E5-2690
E5-2697 v2
E5-2699 v3
E5-2699 v3
E5-2699 v4
INTEL Xeon CPUs
NVIDIA Tesla GPUs
AMD Radeon GPUs
INTEL Xeon Phis

Crucial for Deep Learning



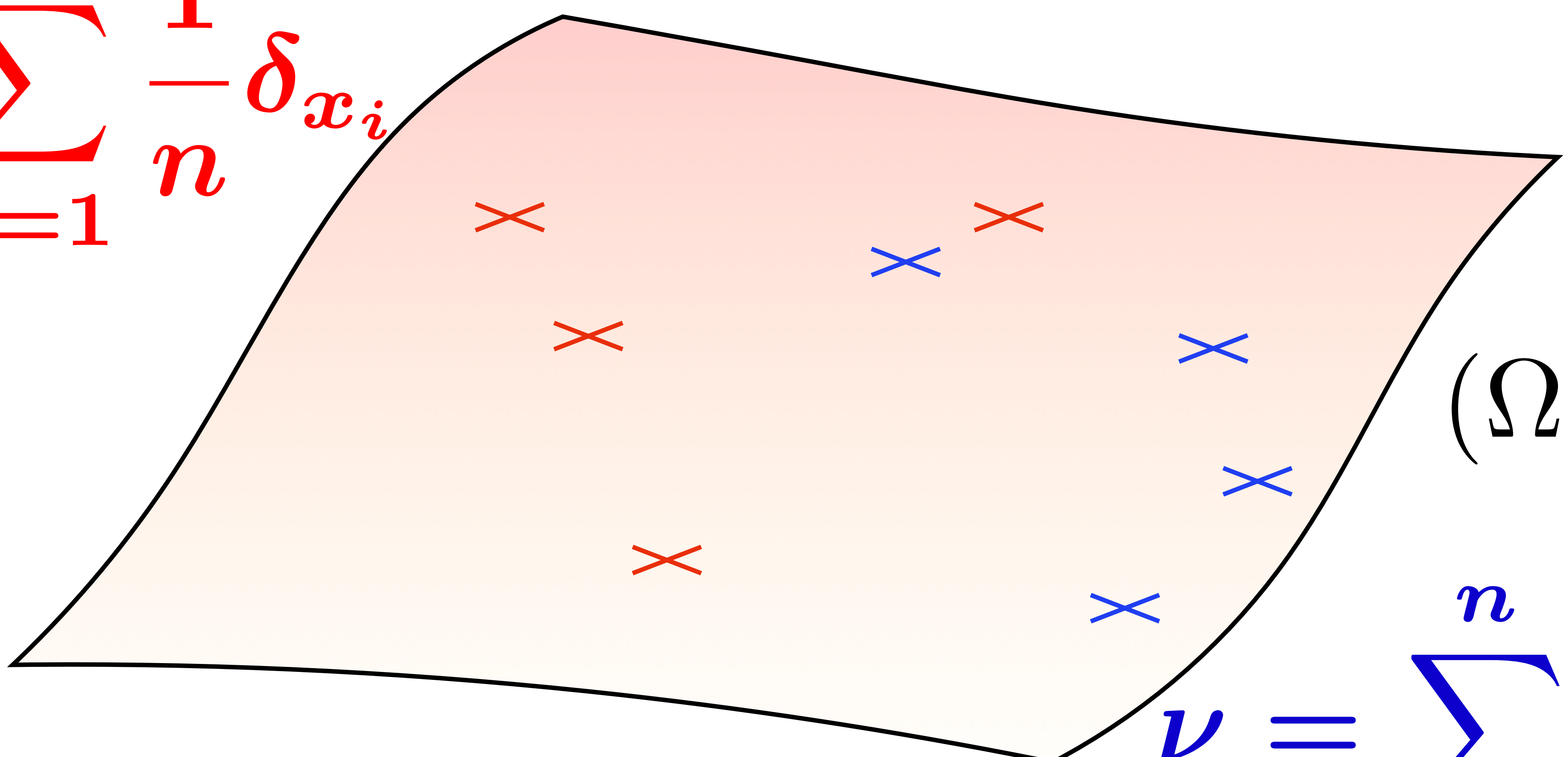
Why?

Multilayer Neural Networks only use element wise operations (hinge, softmax, tanh, sigmoid) and matrix products, exactly those operations that GPU are good for.

A more concrete Math Problem.

Optimal Assignment Problem

$$\mu = \sum_{i=1}^{n} \frac{1}{n} \delta_{x_i}$$



$$(\Omega, D)$$

$$\nu = \sum_{j=1}^{n} \frac{1}{n} \delta_{y_j}$$

A more concrete Math Problem.

Optimal Assignment Problem

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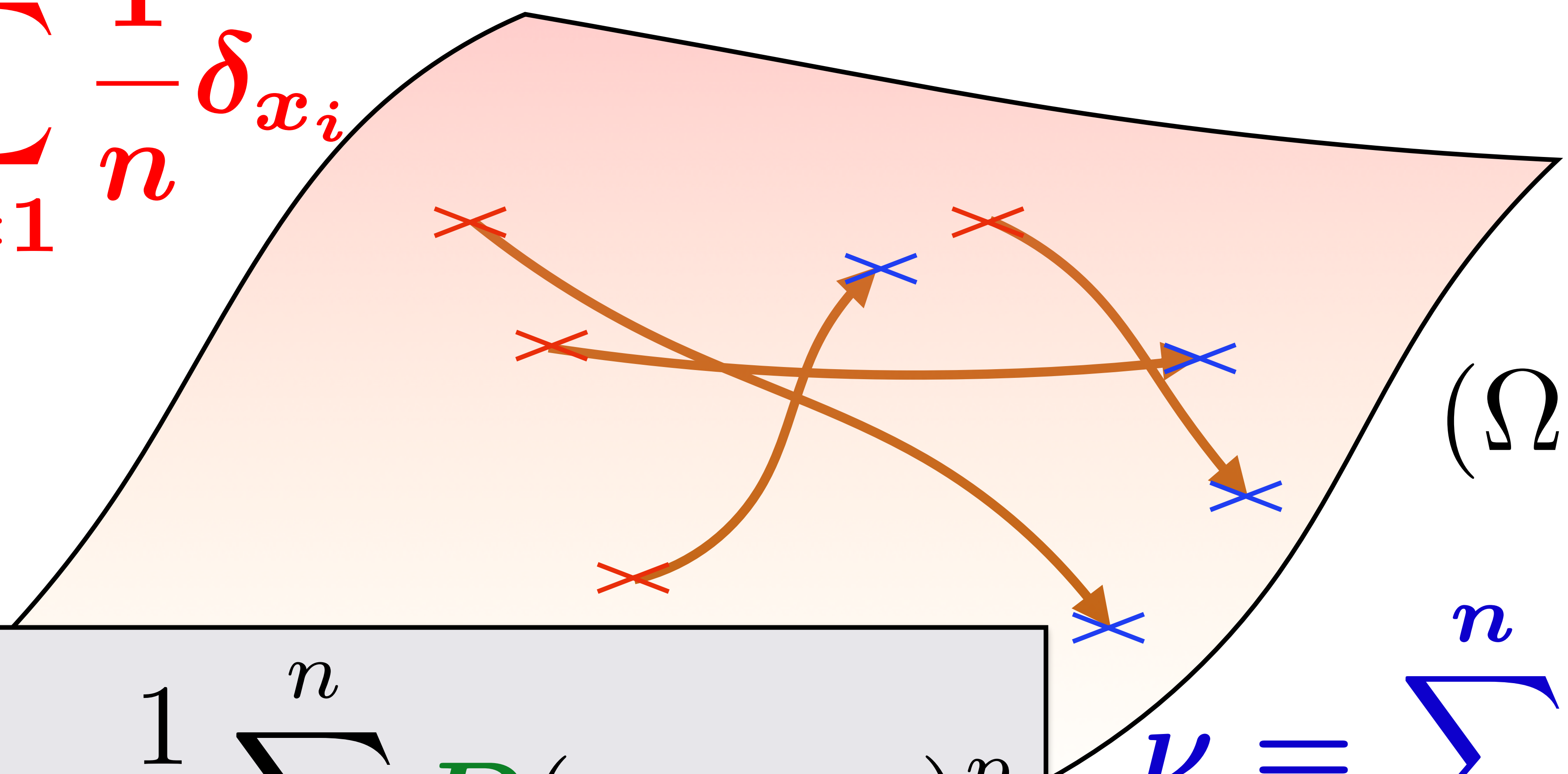


$$(\Omega, D)$$

$$\boldsymbol{\nu} = \sum_{j=1}^{n} \frac{1}{n} \boldsymbol{\delta}_{\boldsymbol{y}_j}$$

$$\boxed{C(\boldsymbol{\sigma}) = \frac{1}{n} \sum_{i=1}^{n} D(\boldsymbol{x}_i, \boldsymbol{y}_{\boldsymbol{\sigma}_i})^p}$$

Optimal Assignment Problem

$$\boxed{\mathrm{OA}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \min_{\boldsymbol{\sigma} \in S_n} C(\boldsymbol{\sigma})}$$

$$M_{\mathbf{XY}} \stackrel{\text{def}}{=} [D(\boldsymbol{x}_i, \boldsymbol{y}_j)^p]_{ij}$$

$$P_{\boldsymbol{\sigma}} = [\mathbf{1}_{\boldsymbol{\sigma}_i = j}/n]_{i,j}$$

$$\boxed{\min_{\boldsymbol{\sigma} \in S_n} C(\boldsymbol{\sigma}) = \min_{\boldsymbol{\sigma} \in S_n} \langle P_{\boldsymbol{\sigma}}, M_{\mathbf{XY}} \rangle}$$

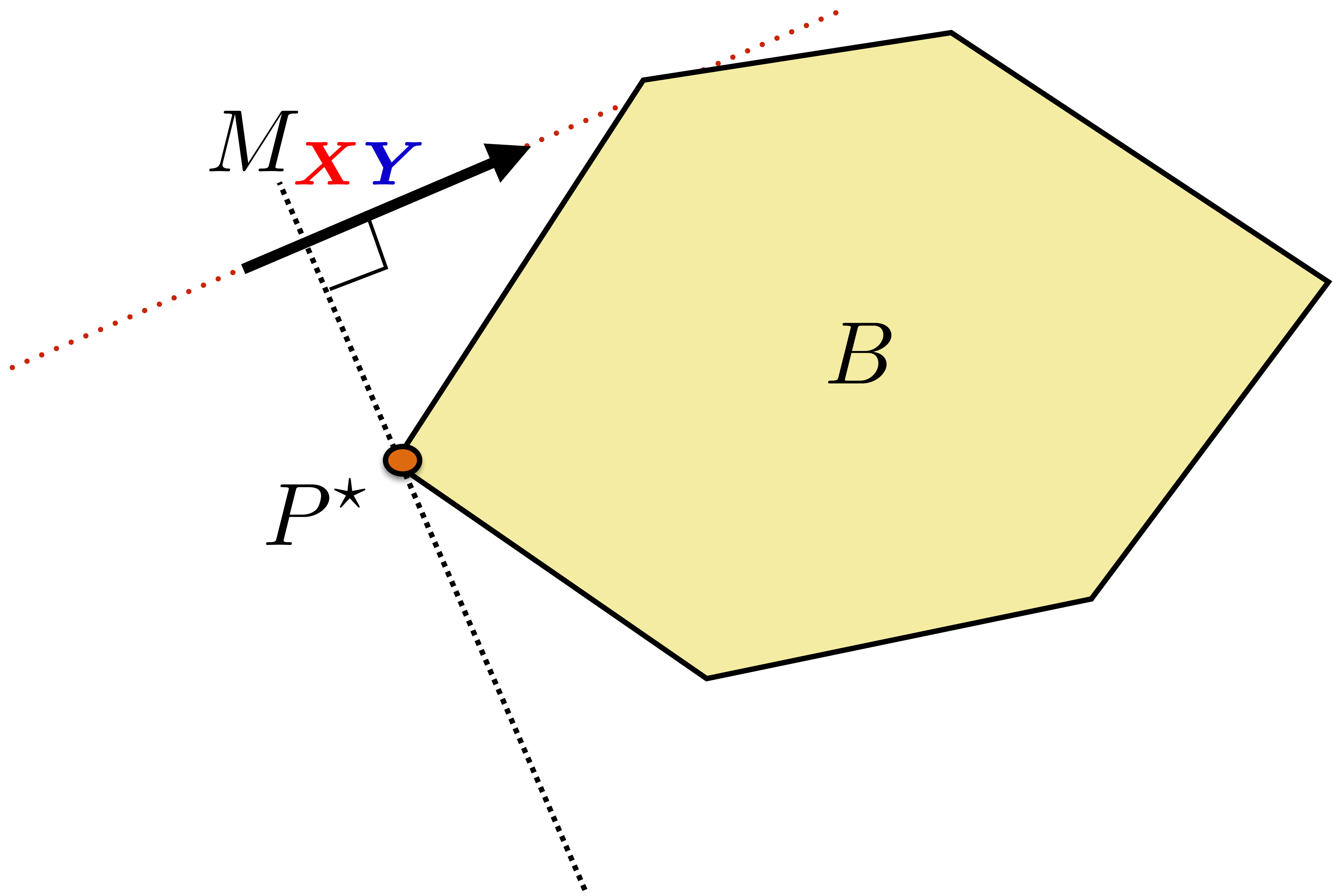
Optimal Assignment Problem

$$\min_{\boldsymbol{\sigma} \in S_n} C(\boldsymbol{\sigma}) = \min_{\boldsymbol{\sigma} \in S_n} \langle P_{\boldsymbol{\sigma}}, M_{\mathbf{X}\mathbf{Y}} \rangle$$

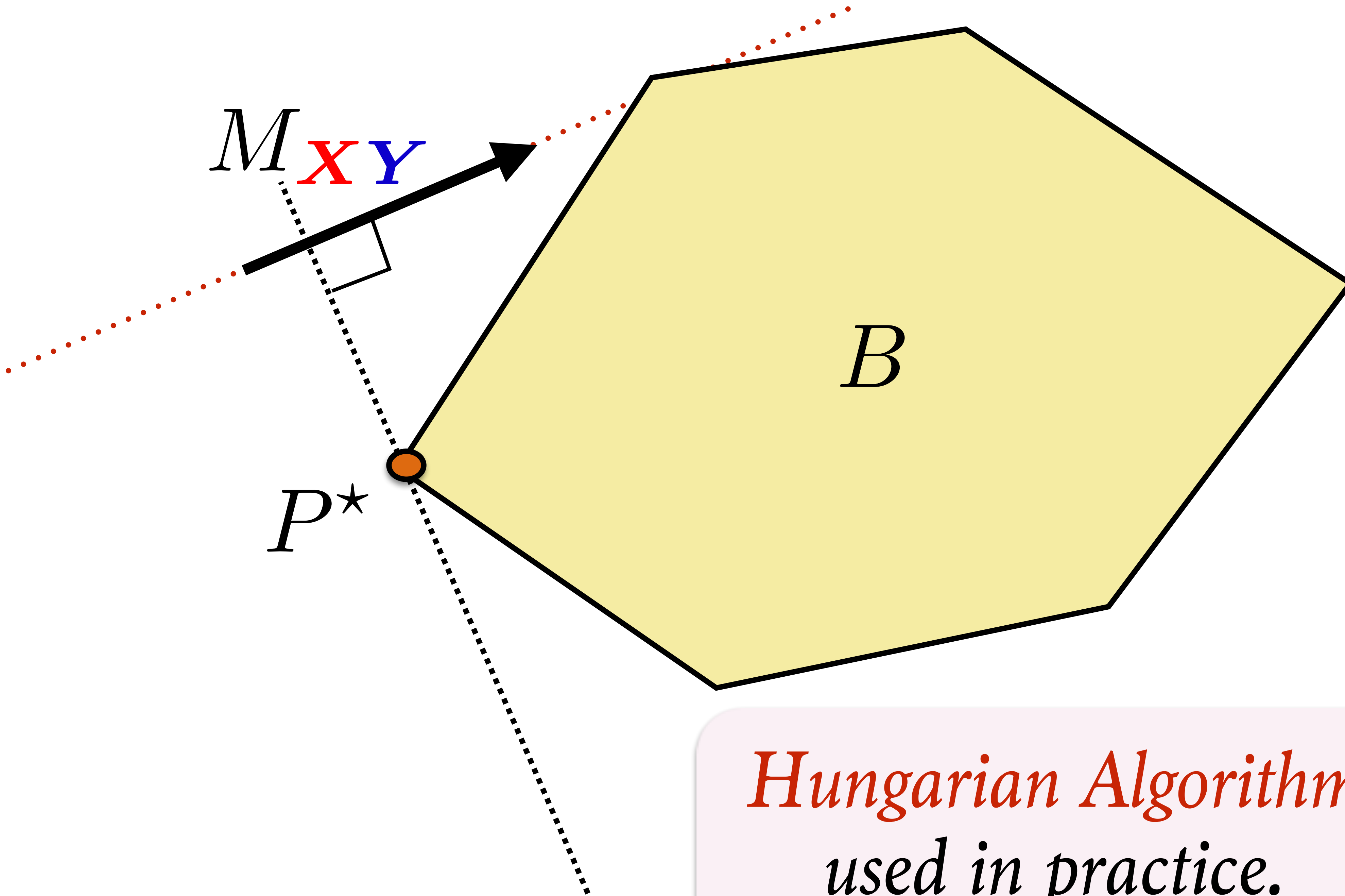
$$B = \left\{ P \in \mathbb{R}_+^{n \times n} | P\mathbf{1} = P^T\mathbf{1} = \frac{\mathbf{1}}{n} \right\}$$

$$\mathrm{OA}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \min_{\boldsymbol{P} \in B} \langle \boldsymbol{P}, M_{\mathbf{X}\mathbf{Y}} \rangle$$

Optimal Assignment



Optimal Assignment



Hungarian Algorithm *used in practice.*

Solving OA using Matrix Products

$$\mathrm{OA}_\gamma(\boldsymbol{\mu}, \boldsymbol{\nu}) = \min_{\boldsymbol{P} \in B} \langle \boldsymbol{P}, M_{\mathbf{XY}} \rangle - \gamma E(\boldsymbol{P})$$

$$E(P) \stackrel{\text{def}}{=} - \sum_{i,j=1}^{n} P_{ij} (\log P_{ij})$$

Solving OA using Matrix Products

$$\mathrm{OA}_\gamma(\boldsymbol{\mu},\boldsymbol{\nu}) = \min_{\boldsymbol{P}\in B}\langle \boldsymbol{P}, M_{\mathbf{X}\mathbf{Y}}\rangle - \gamma E(\boldsymbol{P})$$

$$E(P) \stackrel{\text{def}}{=} -\sum_{i,j=1}^{n} P_{ij}(\log P_{ij})$$

$$L(P,\alpha,\beta) = \sum_{ij} P_{ij}M_{ij} + \gamma P_{ij}\log P_{ij} + \alpha^T(P\mathbf{1}-\mathbf{1}/n) + \beta^T(P^T\mathbf{1}-\mathbf{1}/n)$$

$$\partial L/\partial P_{ij} = M_{ij} + \gamma(\log P_{ij}+1) + \alpha_i + \beta_j$$

$$(\boldsymbol{\partial L/\partial P_{ij}} = 0) \Rightarrow P_{ij} = e^{\frac{\alpha_i}{\gamma}+\frac{1}{2}}\, e^{-\frac{M_{ij}}{\gamma}}\, e^{\frac{\beta_j}{\gamma}+\frac{1}{2}} = \boldsymbol{u_i}\, K_{ij}\boldsymbol{v_j}$$

Solving OA using Matrix Products

$$\mathrm{OA}(\boldsymbol{\mu},\boldsymbol{\nu}) = \min_{\boldsymbol{P}\in B}\langle \boldsymbol{P}, M_{\mathbf{XY}}\rangle$$

Hungarian Algorithm
Cubic complexity

$$\mathrm{OA}_\gamma(\boldsymbol{\mu},\boldsymbol{\nu}) = \min_{\boldsymbol{P}\in B}\langle \boldsymbol{P}, M_{\mathbf{XY}}\rangle - \gamma E(\boldsymbol{P})$$

$$\boldsymbol{P}^* = D(\boldsymbol{u})KD(\boldsymbol{v}); \boldsymbol{u} = \frac{\mathbf{1}}{nK\boldsymbol{v}}, \boldsymbol{v} = \frac{\mathbf{1}}{nK^T\boldsymbol{u}}$$

Automatic Differentiation

Automatic Differentiation

Automatic differentiation:

set of techniques to numerically evaluate the derivative of a function specified by a computer program.

Automatic differentiation is **not**

numerical differentiation

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \qquad \frac{\partial f(\mathbf{x})}{\partial x_i} \approx \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h}$$

symbolic differentiation

$$\frac{d}{dx}(f(x) + g(x)) \rightsquigarrow \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}(f(x)\,g(x)) \rightsquigarrow \left(\frac{d}{dx}f(x)\right) g(x) + f(x)\left(\frac{d}{dx}g(x)\right) .$$

Automatic Differentiation

$$l_1 = x$$
$$l_n + 1 = 4l_n(1 - l_n)$$

$$f(x) = l_4 = 64x(1-x)(1-2x)^2(1-8x+8x^2)^2$$

Source: *Automatic differentiation in machine learning, a survey, Baydin et. al, 2015*

Automatic Differentiation

$l_1 = x$
$l_n + 1 = 4l_n(1 - l_n)$

$f(x) = l_4 = 64x(1-x)(1-2x)^2(1-8x+8x^2)^2$

Manual
Differentiation

$$f'(x) = 128x(1-x)(-8+16x)(1-2x)^2(1-8x+8x^2) + 64(1-x)(1-2x)^2(1-8x+8x^2)^2 - 64x(1-2x)^2(1-8x+8x^2)^2 - 256x(1-x)(1-2x)(1-8x+8x^2)^2$$

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Automatic Differentiation

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Manual Differentiation

$f'(x) = 128x(1-x)(-8+16x)(1-2x)^2(1-8x+8x^2) + 64(1-x)(1-2x)^2(1-8x+8x^2)^2 - 64x(1-2x)^2(1-8x+8x^2)^2 - 256x(1-x)(1-2x)(1-8x+8x^2)^2$

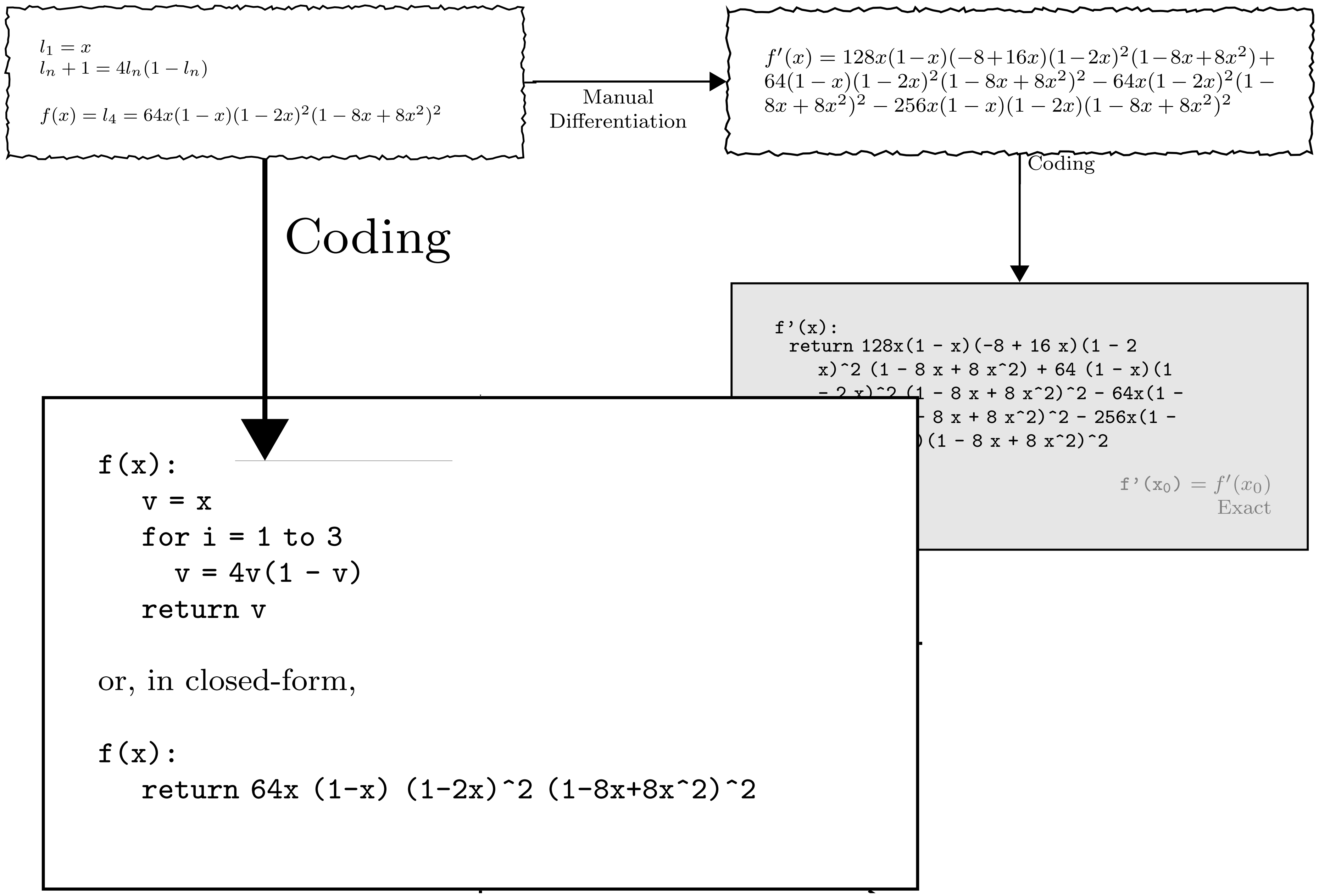
Coding

```
f'(x):
  return 128x(1 - x)(-8 + 16 x)(1 - 2
     x)^2 (1 - 8 x + 8 x^2) + 64 (1 - x)(1
     - 2 x)^2 (1 - 8 x + 8 x^2)^2 - 64x(1 -
     2 x)^2 (1 - 8 x + 8 x^2)^2 - 256x(1 -
     x)(1 - 2 x)(1 - 8 x + 8 x^2)^2
```

f'(x_0) $= f'(x_0)$
Exact

Source: *Automatic differentiation in machine learning, a survey, Baydin et. al, 2015*

Automatic Differentiation



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Automatic Differentiation

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Manual
Differentiation

$f'(x) = 128x(1-x)(-8+16x)(1-2x)^2(1-8x+8x^2) + 64(1-x)(1-2x)^2(1-8x+8x^2)^2 - 64x(1-2x)^2(1-8x+8x^2)^2 - 256x(1-x)(1-2x)(1-8x+8x^2)^2$

Coding

Coding

```
f(x):
  v = x
  for i = 1 to 3
    v = 4v(1 - v)
  return v
```

or, in closed-form,

```
f(x):
  return 64x (1-x) (1-2x)^2 (1-8x+8x^2)^2
```

Symbolic
Differentiation
of the Closed-form

```
f'(x):
 return 128x(1 - x)(-8 + 16 x)(1 - 2
    x)^2 (1 - 8 x + 8 x^2) + 64 (1 - x)(1
    - 2 x)^2 (1 - 8 x + 8 x^2)^2 - 64x(1 -
    2 x)^2 (1 - 8 x + 8 x^2)^2 - 256x(1 -
    x)(1 - 2 x)(1 - 8 x + 8 x^2)^2
```

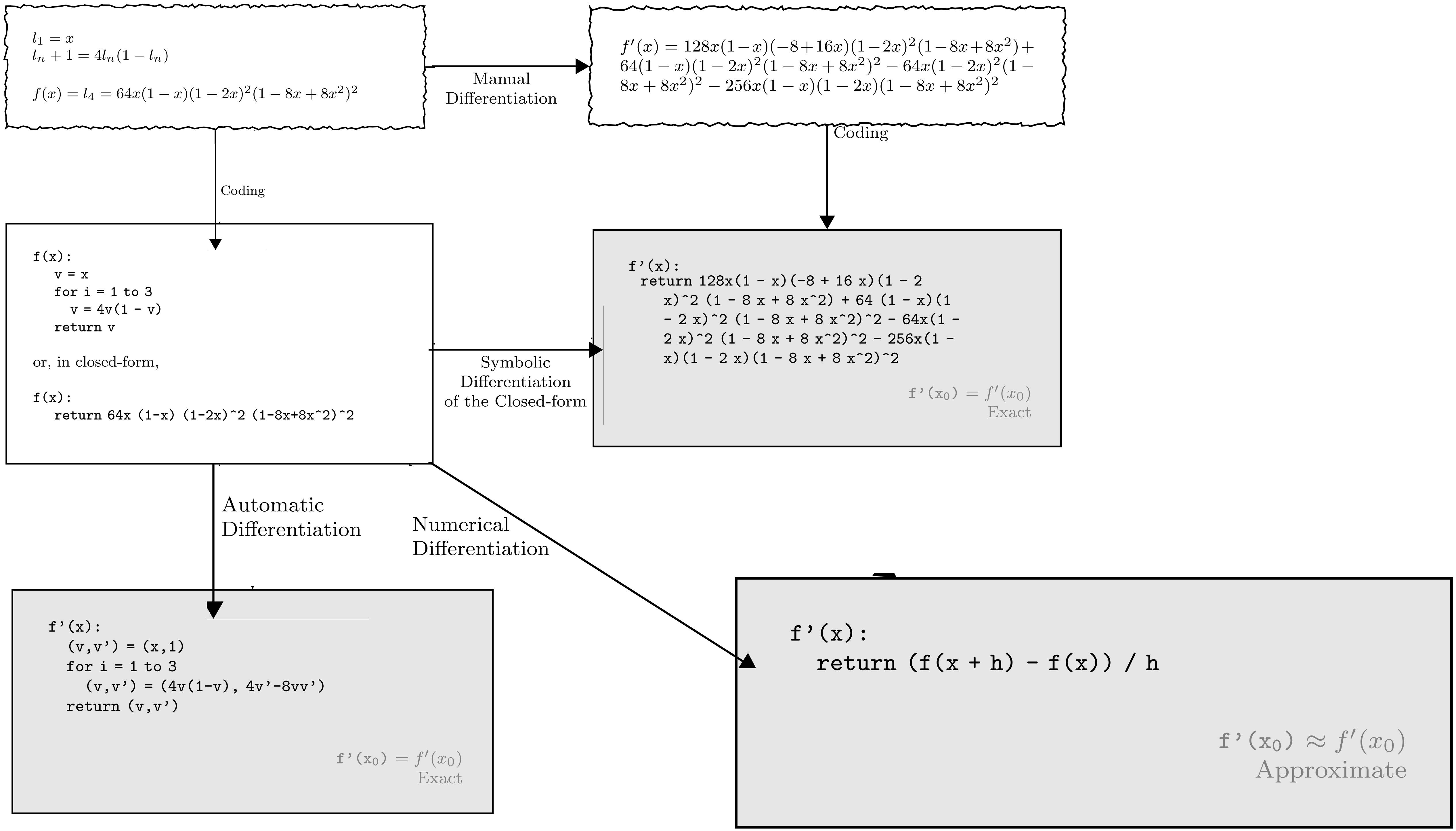
f'(x_0) $= f'(x_0)$
Exact

Automatic
Differentiation

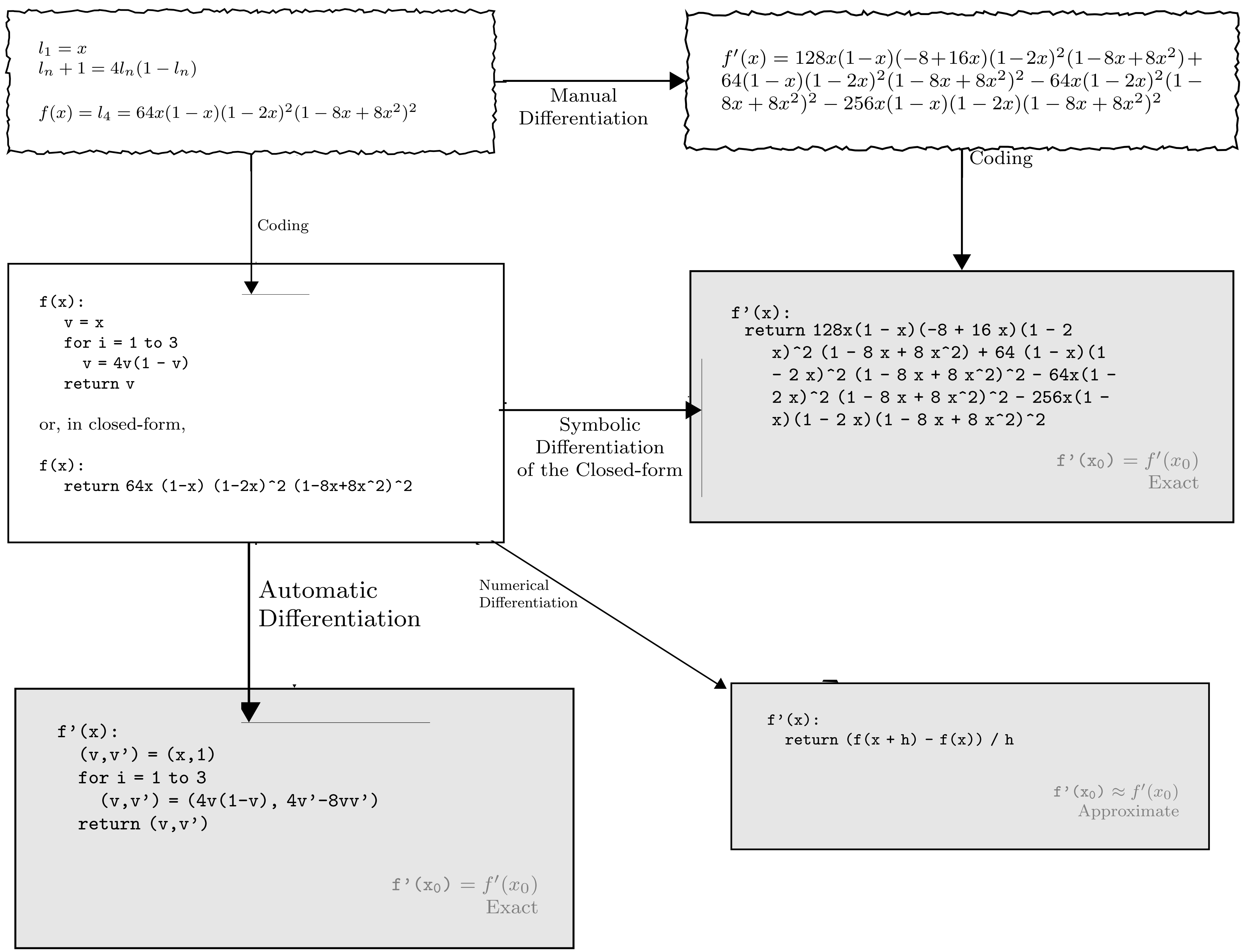
```
f'(x):
  (v,v') = (x,1)
  for i = 1 to 3
    (v,v') = (4v(1-v), 4v'-8vv')
  return (v,v')
```

f'(x_0) $= f'(x_0)$
Exact

Automatic Differentiation



Automatic Differentiation



Automatic Differentiation

n	l_n	$\frac{d}{dx} l_n$	$\frac{d}{dx} l_n$ (Optimized)
1	x	1	1
2	$4x(1-x)$	$4(1-x) - 4x$	$4 - 8x$
3	$16x(1-x)(1-2x)^2$	$16(1-x)(1-2x)^2 - 16x(1-2x)^2 - 64x(1-x)(1-2x)$	$16(1 - 10x + 24x^2 - 16x^3)$
4	$64x(1-x)(1-2x)^2(1-8x+8x^2)^2$	$128x(1-x)(-8+16x)(1-2x)^2(1-8x+8x^2) + 64(1-x)(1-2x)^2(1-8x+8x^2)^2 - 64x(1-2x)^2(1-8x+8x^2)^2 - 256x(1-x)(1-2x)(1-8x+8x^2)^2$	$64(1-42x+504x^2-2640x^3+7040x^4-9984x^5+7168x^6-2048x^7)$

Automatic Differentiation

Computer code for $f(x_1, x_2) = x_1 x_2 + \sin(x_1)$ might read

Original program	Dual program
$w_1 = x_1$	$\dot{w}_1 = 0$
$w_2 = x_2$	$\dot{w}_2 = 1$
$w_3 = w_1 w_2$	$\dot{w}_3 = \dot{w}_1 w_2 + w_1 \dot{w}_2 = 0 \cdot x_2 + x_1 \cdot 1 = x_1$
$w_4 = \sin(w_1)$	$\dot{w}_4 = \cos(w_1)\dot{w}_1 = \cos(x_1) \cdot 0 = 0$
$w_5 = w_3 + w_4$	$\dot{w}_5 = \dot{w}_3 + \dot{w}_4 = x_1 + 0 = x_1$

and

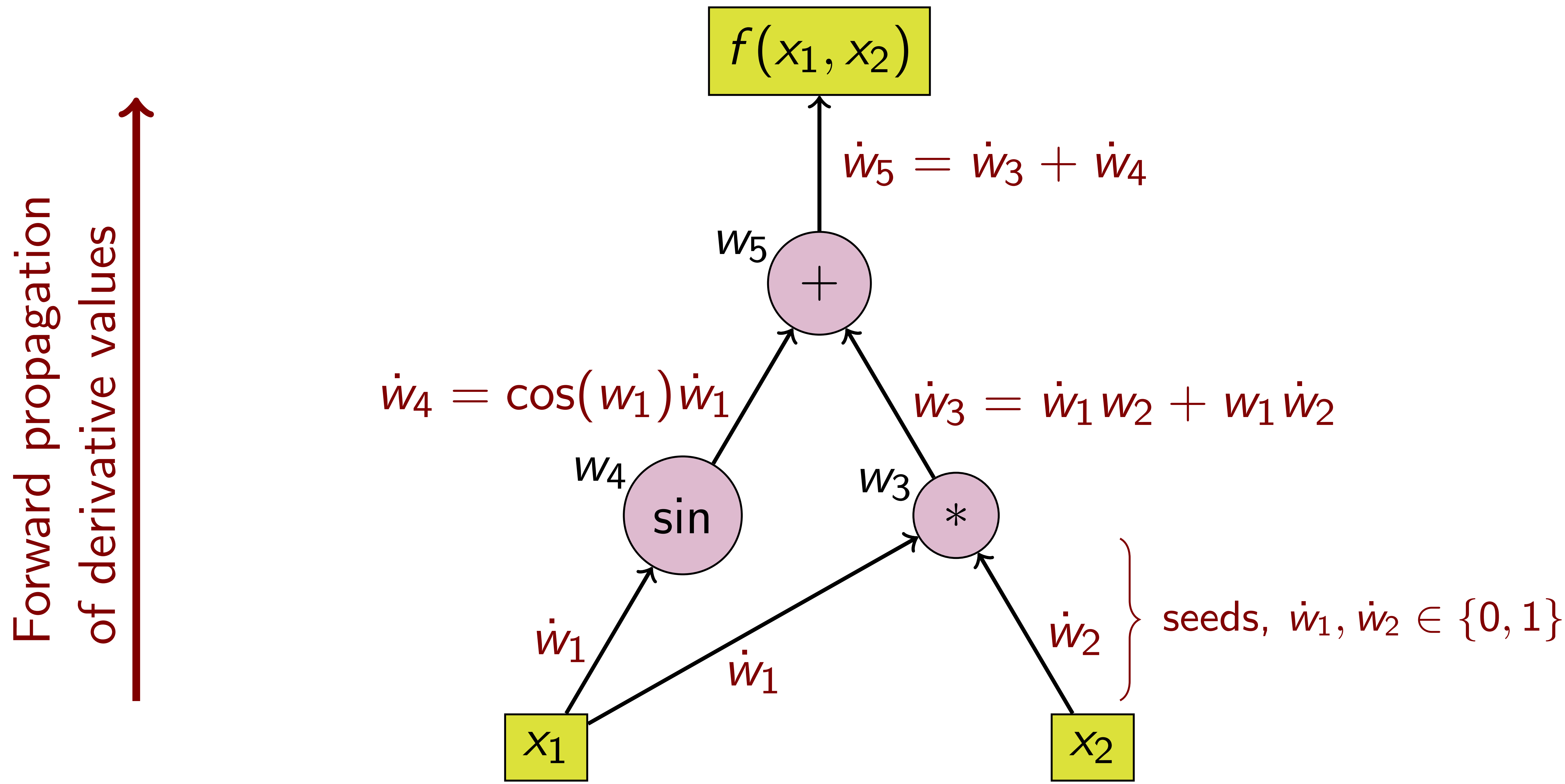
$$\frac{\partial f}{\partial x_2} = x_1$$

The chain rule

$$\frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial w_5} \frac{\partial w_5}{\partial w_3} \frac{\partial w_3}{\partial w_2} \frac{\partial w_2}{\partial x_2}$$

ensures that we can *propagate* the dual components throughout the computation.

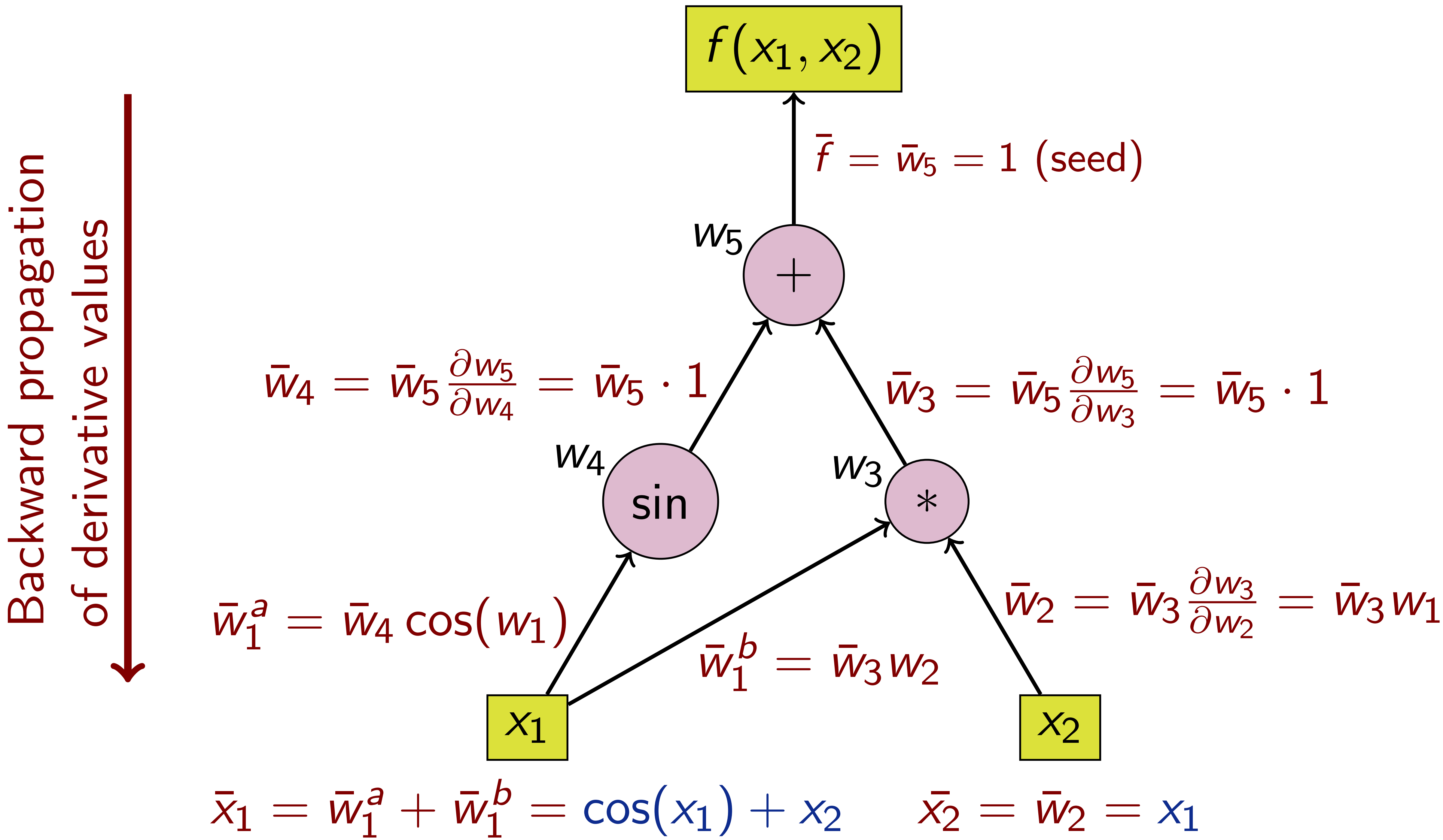
Automatic Differentiation



$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial w_1}\frac{\partial w_1}{\partial x} = \frac{\partial y}{\partial w_1}\left(\frac{\partial w_1}{\partial w_2}\frac{\partial w_2}{\partial x}\right) = \frac{\partial y}{\partial w_1}\left(\frac{\partial w_1}{\partial w_2}\left(\frac{\partial w_2}{\partial w_3}\frac{\partial w_3}{\partial x}\right)\right) = \cdots$$

Source: Havard Berland, NTNU

Automatic Differentiation



$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial w_1}\frac{\partial w_1}{\partial x} = \left(\frac{\partial y}{\partial w_2}\frac{\partial w_2}{\partial w_1}\right)\frac{\partial w_1}{\partial x} = \left(\left(\frac{\partial y}{\partial w_3}\frac{\partial w_3}{\partial w_2}\right)\frac{\partial w_2}{\partial w_1}\right)\frac{\partial w_1}{\partial x} = \cdots$$

Source: Havard Berland, NTNU

Automatic Differentiation

Given $F\colon \mathbf{R}^n \mapsto \mathbf{R}^m$ and the Jacobian $J = DF(\mathbf{x}) \in \mathbf{R}^{m \times n}$.

$$J = DF(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

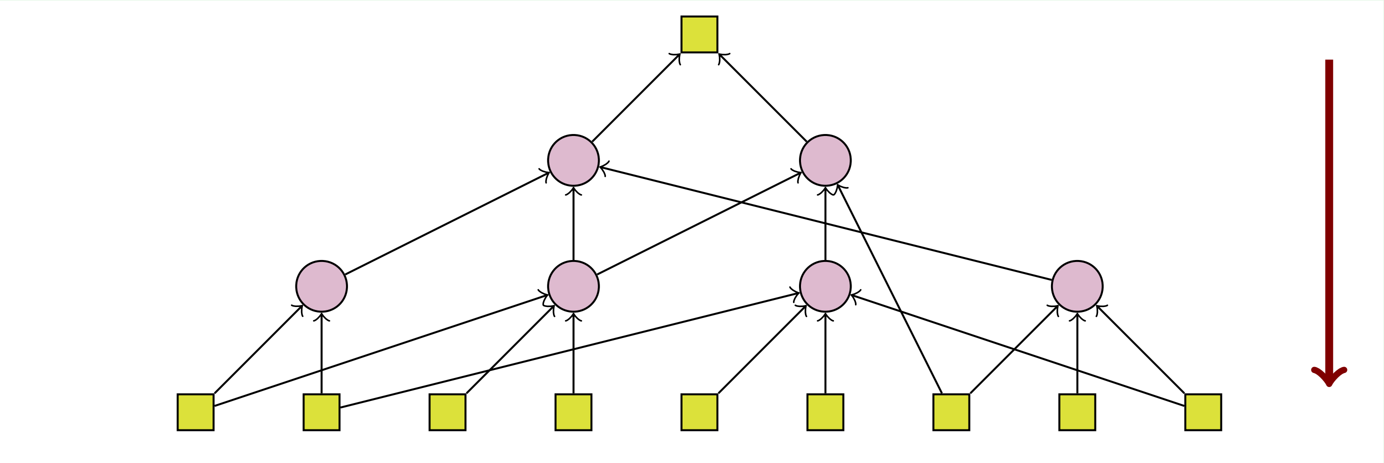
Backward sweep can compute
one column

Forward sweep can compute
one column

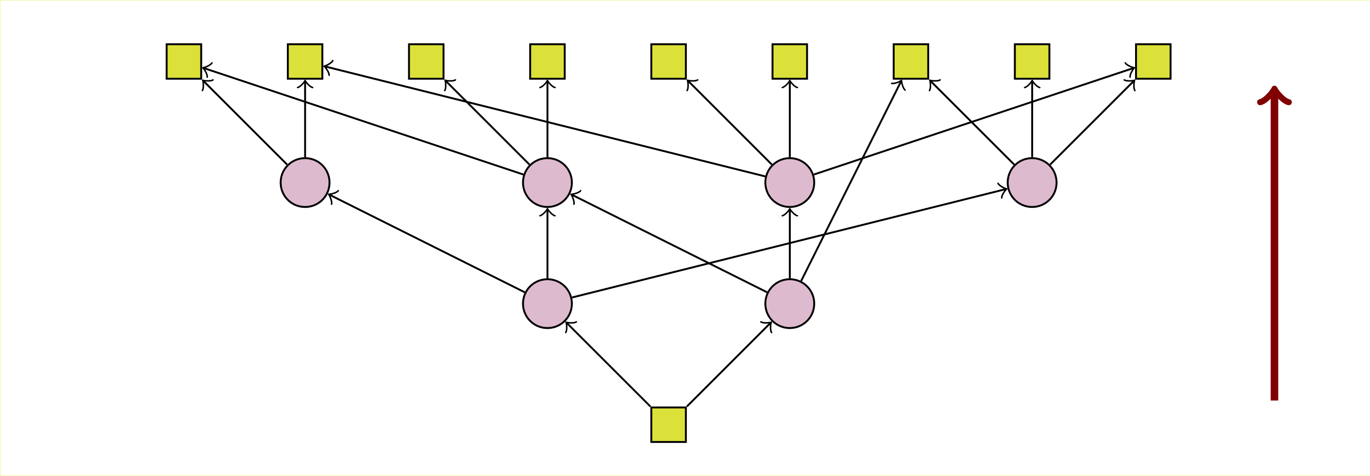
Source: Havard Berland, NTNU

Automatic Differentiation

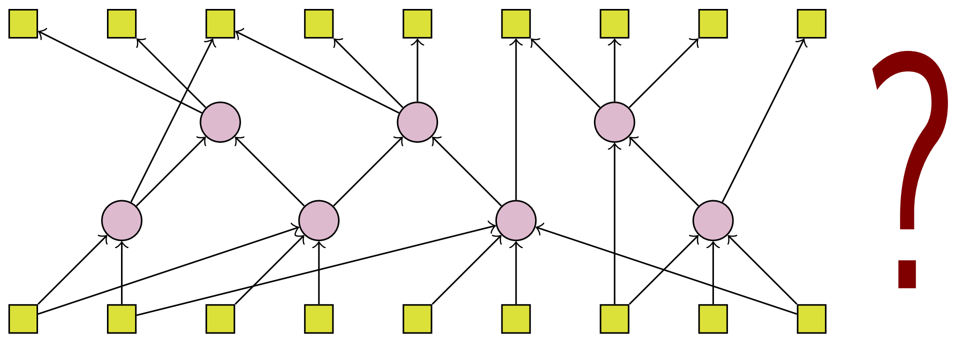
Reverse mode suitable for $F : \mathbb{R}^p \mapsto \mathbb{R}$



Forward mode suitable for $F : \mathbb{R} \mapsto \mathbb{R}^p$



$$F : \mathbb{R}^d \mapsto \mathbb{R}^p \ ?$$



Source: Havard Berland, NTNU

Distributed Optimization

Primal Methods

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n} \sum_{i=1}^{n} l_i(\boldsymbol{\theta})$$

$$\boxed{\text{We want to approximate } \nabla \tfrac{1}{n} \textstyle\sum_{i=1}^{n} l_i(\boldsymbol{\theta})}$$

Primal Methods

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n} \sum_{i=1}^{n} l_i(\boldsymbol{\theta})$$

$$\boxed{\text{We want to approximate } \nabla \tfrac{1}{n} \textstyle\sum_{i=1}^{n} l_i(\boldsymbol{\theta})}$$

$$\mathbb{E}_{i \sim \mathrm{unif}\{1,\ldots,\mathrm{n}\}}[\nabla l_i(\boldsymbol{\theta})] = \frac{1}{n} \sum_{i} \nabla l_i(\boldsymbol{\theta}) = \nabla \mathcal{L}(\boldsymbol{\theta})$$

Stochastic approaches mentioned in F. Bach's talk.

Primal Methods

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n} \sum_{i=1}^{n} l_i(\boldsymbol{\theta})$$

$\boxed{\text{We want to approximate } \nabla \frac{1}{n} \sum_{i=1}^{n} l_i(\boldsymbol{\theta})}$

$$\mathbb{E}_{i \sim \mathrm{unif}\{1,\ldots,\mathrm{n}\}}[\nabla l_i(\boldsymbol{\theta})] = \frac{1}{n} \sum_{i} \nabla l_i(\boldsymbol{\theta}) = \nabla \mathcal{L}(\boldsymbol{\theta})$$

Primal Methods

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n} \sum_{i=1}^{n} l_i(\boldsymbol{\theta})$$

$n \approx \infty$

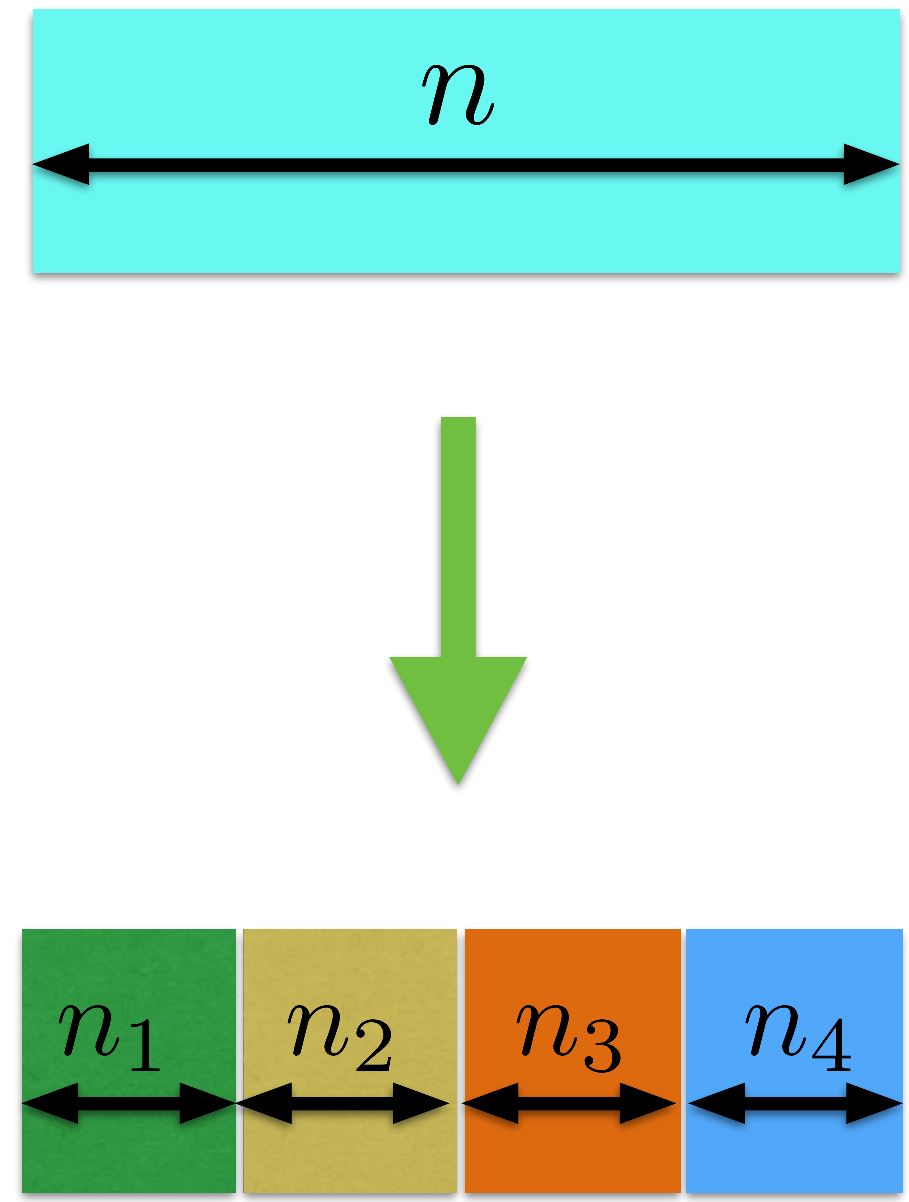
We want to approximate $\nabla \frac{1}{n} \sum_{i=1}^{n} l_i(\boldsymbol{\theta})$

$$\mathbb{E}_{i \sim \text{unif}\{1,\ldots,\text{n}\}}[\nabla l_i(\boldsymbol{\theta})] = \frac{1}{n} \sum_{i} \nabla l_i(\boldsymbol{\theta}) = \nabla \mathcal{L}(\boldsymbol{\theta})$$

Distributed Primal Methods

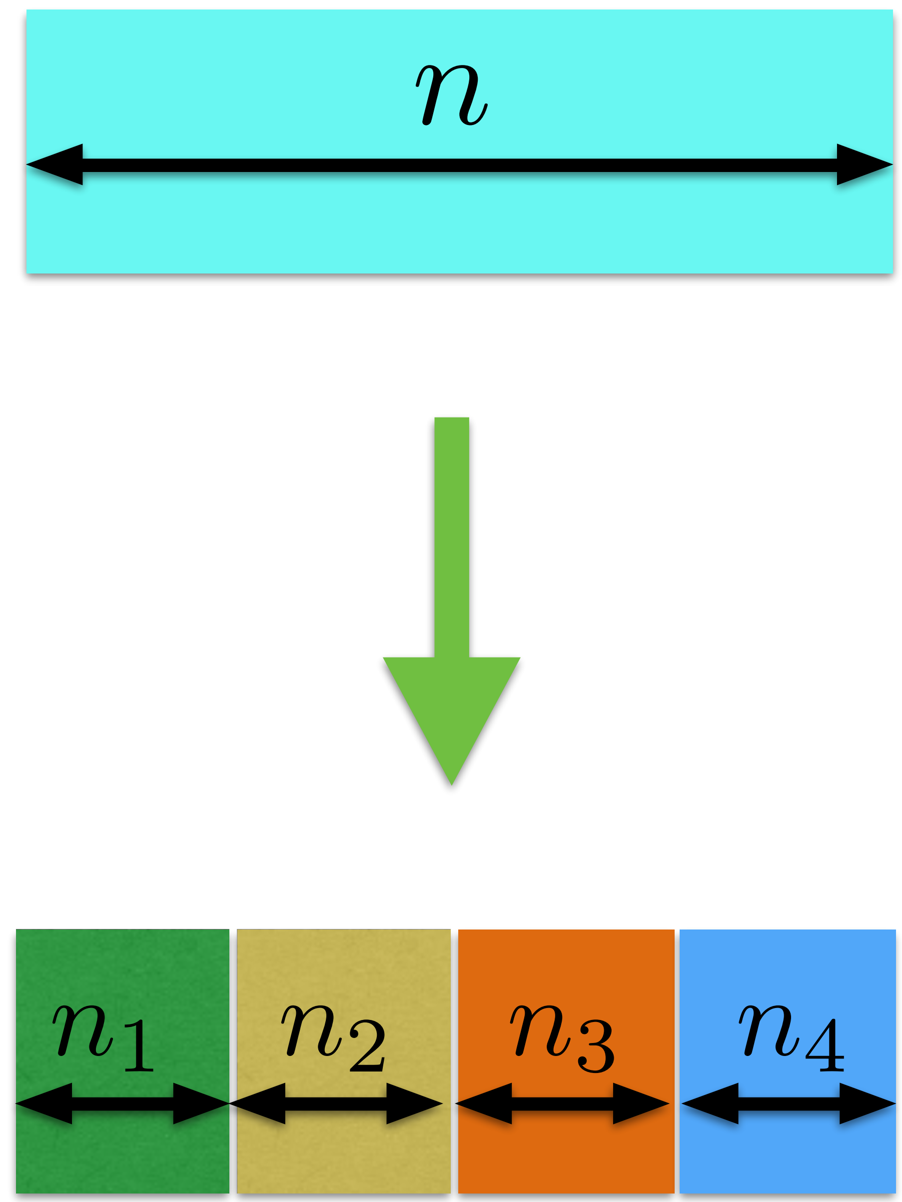
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n} \sum_{i=1}^{n} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \sum_{j=1}^{N} \left(\frac{1}{n_j} \sum_{i \in I_j} l_i(\boldsymbol{\theta}) \right)$$



Distributed Primal Methods

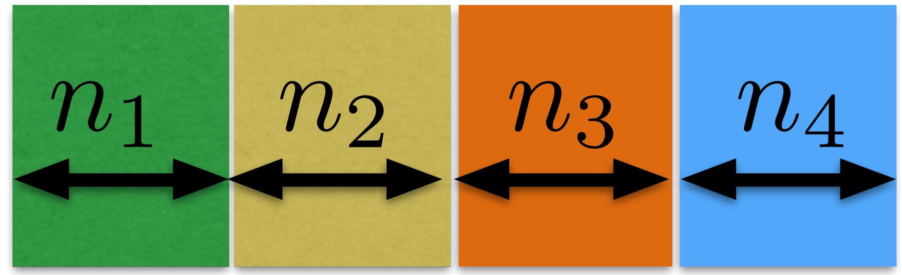
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n} \sum_{i=1}^{n} l_i(\boldsymbol{\theta})$$



$n \approx \infty$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \sum_{j=1}^{N} \left(\frac{1}{n_j} \sum_{i \in I_j} l_i(\boldsymbol{\theta}) \right)$$

Distributed Primal Methods



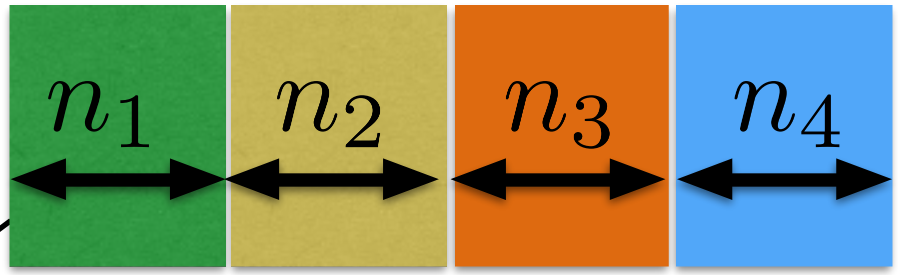
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{1}}} \sum_{i \in I_{\mathbf{1}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{2}}} \sum_{i \in I_{\mathbf{2}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{3}}} \sum_{i \in I_{\mathbf{3}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{4}}} \sum_{i \in I_{\mathbf{4}}} l_i(\boldsymbol{\theta})$$

Distributed Primal Methods



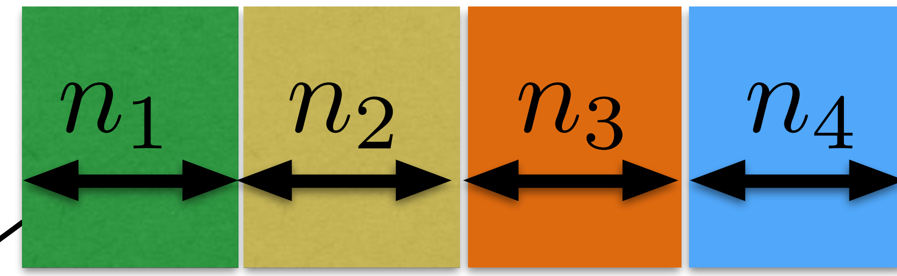
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{1}}} \sum_{i \in I_{\mathbf{1}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{2}}} \sum_{i \in I_{\mathbf{2}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{3}}} \sum_{i \in I_{\mathbf{3}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{4}}} \sum_{i \in I_{\mathbf{4}}} l_i(\boldsymbol{\theta})$$

Distributed Primal Methods



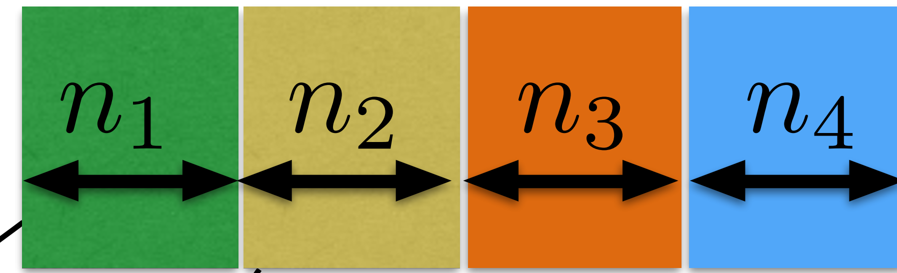
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_{\mathbf{1}}} \sum_{i \in I_{\mathbf{1}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_{\mathbf{2}}} \sum_{i \in I_{\mathbf{2}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_{\mathbf{3}}} \sum_{i \in I_{\mathbf{3}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_{\mathbf{4}}} \sum_{i \in I_{\mathbf{4}}} l_i(\boldsymbol{\theta})$$

Distributed Primal Methods



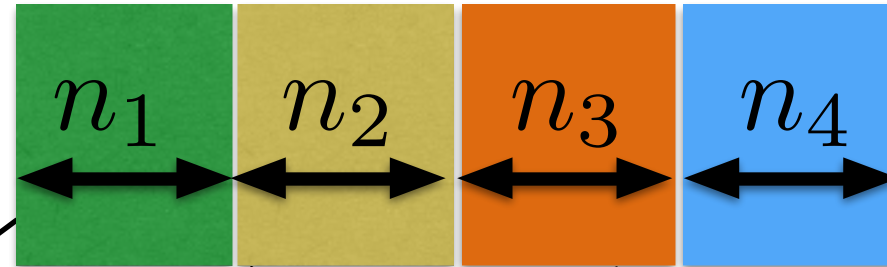
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_{\mathbf{1}}} \sum_{i \in I_{\mathbf{1}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_{\mathbf{2}}} \sum_{i \in I_{\mathbf{2}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_{\mathbf{3}}} \sum_{i \in I_{\mathbf{3}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_{\mathbf{4}}} \sum_{i \in I_{\mathbf{4}}} l_i(\boldsymbol{\theta})$$

Distributed Primal Methods



$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{1}}} \sum_{i \in I_{\mathbf{1}}} l_i(\boldsymbol{\theta})$$

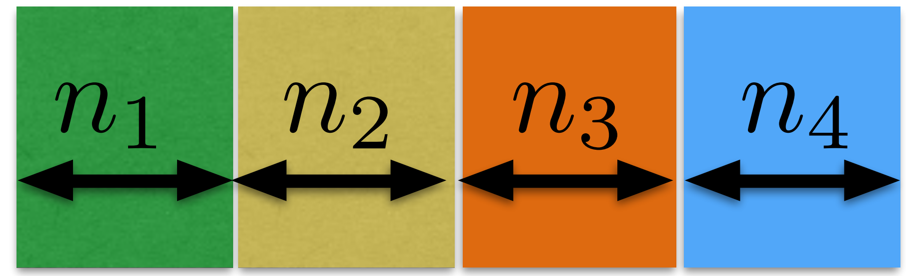
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{2}}} \sum_{i \in I_{\mathbf{2}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{3}}} \sum_{i \in I_{\mathbf{3}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{4}}} \sum_{i \in I_{\mathbf{4}}} l_i(\boldsymbol{\theta})$$

Distributed Primal Methods

100% Parallel



$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_1} \sum_{i\in I_1} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_2} \sum_{i\in I_2} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_3} \sum_{i\in I_3} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_4} \sum_{i\in I_4} l_i(\boldsymbol{\theta})$$

$\boxed{\boldsymbol{\theta}_1^*}$ $\boxed{\boldsymbol{\theta}_2^*}$ $\boxed{\boldsymbol{\theta}_3^*}$ $\boxed{\boldsymbol{\theta}_4^*}$

Distributed Primal Methods

100% Parallel

n_1 n_2 n_3 n_4

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{1}}} \sum_{i \in I_{\mathbf{1}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{2}}} \sum_{i \in I_{\mathbf{2}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{3}}} \sum_{i \in I_{\mathbf{3}}} l_i(\boldsymbol{\theta})$$

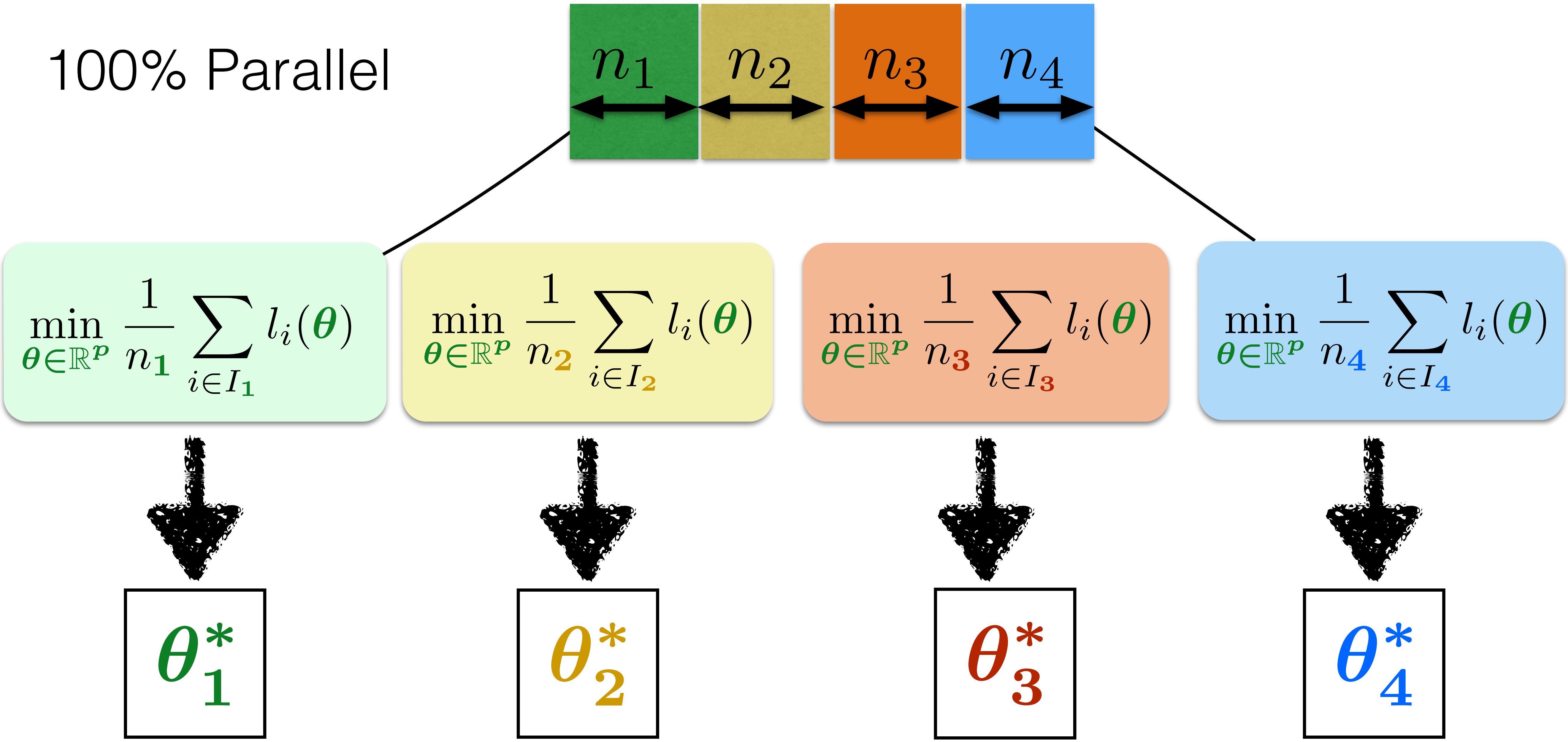
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{4}}} \sum_{i \in I_{\mathbf{4}}} l_i(\boldsymbol{\theta})$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$\boxed{\boldsymbol{\theta}_{\mathbf{1}}^{*}}$ $\boxed{\boldsymbol{\theta}_{\mathbf{2}}^{*}}$ $\boxed{\boldsymbol{\theta}_{\mathbf{3}}^{*}}$ $\boxed{\boldsymbol{\theta}_{\mathbf{4}}^{*}}$

Distributed Primal Methods

100% Parallel



Distributed Primal Methods

100% Parallel

n_1 n_2 n_3 n_4

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{1}}} \sum_{i \in I_{\mathbf{1}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{2}}} \sum_{i \in I_{\mathbf{2}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{3}}} \sum_{i \in I_{\mathbf{3}}} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \frac{1}{n_{\mathbf{4}}} \sum_{i \in I_{\mathbf{4}}} l_i(\boldsymbol{\theta})$$

$\boxed{\boldsymbol{\theta}^*_{\mathbf{1}}}$ $\boxed{\boldsymbol{\theta}^*_{\mathbf{2}}}$ $\boxed{\boldsymbol{\theta}^*_{\mathbf{3}}}$ $\boxed{\boldsymbol{\theta}^*_{\mathbf{4}}}$

Distributed Primal Methods

100% Parallel

n_1 n_2 n_3 n_4

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_1}\sum_{i\in I_1} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_2}\sum_{i\in I_2} l_i(\boldsymbol{\theta})$$

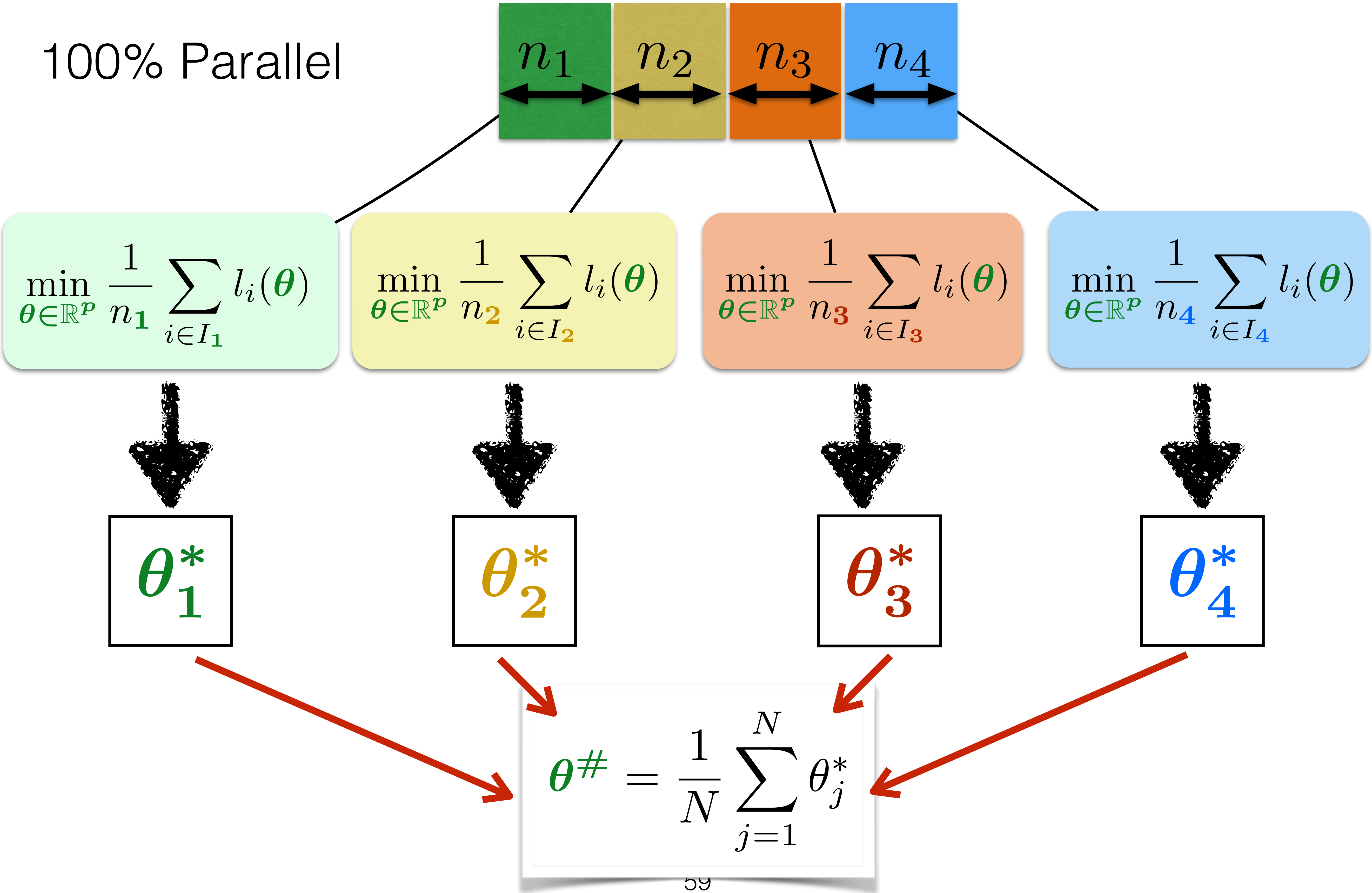
$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_3}\sum_{i\in I_3} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_4}\sum_{i\in I_4} l_i(\boldsymbol{\theta})$$

$\boxed{\boldsymbol{\theta}_1^*}$ $\boxed{\boldsymbol{\theta}_2^*}$ $\boxed{\boldsymbol{\theta}_3^*}$ $\boxed{\boldsymbol{\theta}_4^*}$

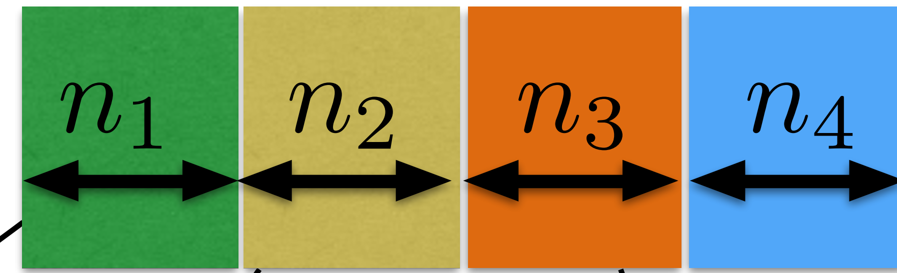
Distributed Primal Methods

100% Parallel



Distributed Primal Methods

Only gradient parallel



$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_{\mathbf{1}}} \sum_{i\in I_{\mathbf{1}}} l_i(\boldsymbol{\theta}) \qquad \min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_{\mathbf{2}}} \sum_{i\in I_{\mathbf{2}}} l_i(\boldsymbol{\theta}) \qquad \min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_{\mathbf{3}}} \sum_{i\in I_{\mathbf{3}}} l_i(\boldsymbol{\theta}) \qquad \min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_{\mathbf{4}}} \sum_{i\in I_{\mathbf{4}}} l_i(\boldsymbol{\theta})$$

$$\boldsymbol{\nabla}_{\mathbf{1}}(\boldsymbol{\theta}_{\mathbf{0}}) \qquad \boldsymbol{\nabla}_{\mathbf{2}}(\boldsymbol{\theta}_{\mathbf{0}}) \qquad \boldsymbol{\nabla}_{\mathbf{3}}(\boldsymbol{\theta}_{\mathbf{0}}) \qquad \boldsymbol{\nabla}_{\mathbf{4}}(\boldsymbol{\theta}_{\mathbf{0}})$$

Communication cost!!!

How can we incorporate regularizer?

Distributed Primal Methods

Only gradient parallel

$n_1 \quad n_2 \quad n_3 \quad n_4$

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_1}\sum_{i\in I_1} l_i(\boldsymbol{\theta}) \qquad \min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_2}\sum_{i\in I_2} l_i(\boldsymbol{\theta}) \qquad \min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_3}\sum_{i\in I_3} l_i(\boldsymbol{\theta}) \qquad \min_{\boldsymbol{\theta}\in\mathbb{R}^p} \frac{1}{n_4}\sum_{i\in I_4} l_i(\boldsymbol{\theta})$$

$$\boldsymbol{\nabla}_1(\boldsymbol{\theta}_0) \qquad \boldsymbol{\nabla}_2(\boldsymbol{\theta}_0) \qquad \boldsymbol{\nabla}_3(\boldsymbol{\theta}_0) \qquad \boldsymbol{\nabla}_4(\boldsymbol{\theta}_0)$$

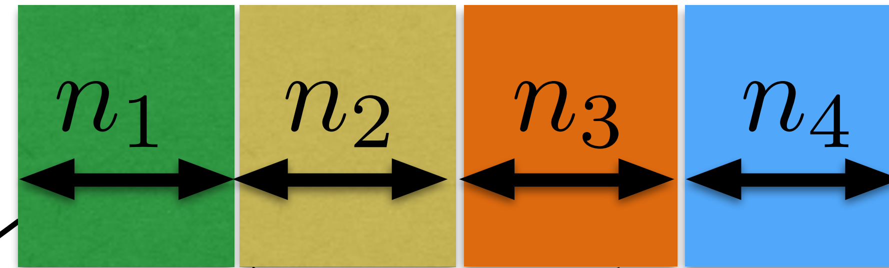
$$\nabla(\boldsymbol{\theta}_0) = \frac{n_1\boldsymbol{\nabla}_1 + n_2\boldsymbol{\nabla}_2 + n_3\boldsymbol{\nabla}_3 + n_4\boldsymbol{\nabla}_4}{n}$$

Communication cost!!!

How can we incorporate regularizer?

Distributed Primal Methods

Only gradient parallel



$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_1} \sum_{i \in I_1} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_2} \sum_{i \in I_2} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_3} \sum_{i \in I_3} l_i(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n_4} \sum_{i \in I_4} l_i(\boldsymbol{\theta})$$

$$\boldsymbol{\theta}_1 = \nabla(\boldsymbol{\theta}_0) - \rho \nabla(\boldsymbol{\theta}_0)$$

Communication cost!!!

How can we incorporate regularizer?

ADMM & Splitting Methods

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \sum_{j=1}^{N} \left(\frac{1}{n_j} \sum_{i \in I_j} l_i(\boldsymbol{\theta}) \right) = \min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \sum_{j=1}^{N} f_j(\boldsymbol{\theta})$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{p}}} \sum_{j=1}^{N} f_j(\boldsymbol{\theta}) = \min_{\substack{\boldsymbol{\theta}_{\boldsymbol{1}}, \ldots, \boldsymbol{\theta}_{\boldsymbol{N}} \in \mathbb{R}^{\boldsymbol{p}} \\ \boldsymbol{\theta}_{\boldsymbol{1}} = \boldsymbol{\theta}_{\boldsymbol{2}} = \cdots = \boldsymbol{\theta}_{\boldsymbol{N}}}} \sum_{j=1}^{N} f_j(\boldsymbol{\theta}_{\boldsymbol{j}})$$

ADMM & Splitting Methods

$$\boxed{\min_{\substack{\boldsymbol{\theta}_{\mathbf{1}},\ldots,\boldsymbol{\theta}_{\boldsymbol{N}} \in \mathbb{R}^{\boldsymbol{p}} \\ \boldsymbol{\rho} = \boldsymbol{\theta}_{\mathbf{1}} = \boldsymbol{\theta}_{\mathbf{2}} = \cdots = \boldsymbol{\theta}_{\boldsymbol{N}}}} \sum_{j=1}^{N} f_j(\boldsymbol{\theta}_{\boldsymbol{j}}) + \boldsymbol{\psi}(\boldsymbol{\rho})}$$

ADMM & Splitting Methods

The generic splitting problem we will address:

$$\boxed{\min_{\substack{\boldsymbol{\theta}_{\mathbf{1}},\ldots,\boldsymbol{\theta}_{\boldsymbol{N}} \in \mathbb{R}^{\boldsymbol{p}} \\ \boldsymbol{\rho}=\boldsymbol{\theta}_{\mathbf{1}}=\boldsymbol{\theta}_{\mathbf{2}}=\cdots=\boldsymbol{\theta}_{\boldsymbol{N}}}} \sum_{j=1}^{N} f_j(\boldsymbol{\theta}_{\boldsymbol{j}}) + \boldsymbol{\psi}(\boldsymbol{\rho})}$$

ADMM & Splitting Methods

$$\boxed{\min_{\substack{\boldsymbol{\theta}_{\boldsymbol{1}},\ldots,\boldsymbol{\theta}_{\boldsymbol{N}} \in \mathbb{R}^{\boldsymbol{p}} \\ \boldsymbol{\rho}=\boldsymbol{\theta}_{\boldsymbol{1}}=\boldsymbol{\theta}_{\boldsymbol{2}}=\cdots=\boldsymbol{\theta}_{\boldsymbol{N}}}} \sum_{j=1}^{N} f_j(\boldsymbol{\theta}_{\boldsymbol{j}}) + \boldsymbol{\psi}(\boldsymbol{\rho})}$$

ADMM & Splitting Methods

repeat for $t = 0, \dots, T$

$$\boldsymbol{\theta}_{\mathbf{1}}^{t+1} = \operatorname*{argmin}_{\theta} f_{\mathbf{1}}(\theta) + \frac{\tau}{2}\|\theta - \boldsymbol{\rho}^t + u_{\mathbf{1}}^t\|^2$$

$$\vdots \quad \vdots$$

$$\boldsymbol{\theta}_{\boldsymbol{N}}^{t+1} = \operatorname*{argmin}_{\theta} f_{\boldsymbol{N}}(\theta) + \frac{\tau}{2}\|\theta - \boldsymbol{\rho}^t + u_{\boldsymbol{N}}^t\|^2$$

$$\boldsymbol{\rho}^{t+1} = \operatorname*{argmin}_{\theta} \psi(\theta) + (N\tau/2)\|\theta - \boldsymbol{\theta}^{t+1} - \bar{u}^t\|^2$$

$$u_{\boldsymbol{i}}^{t+1} = u_{\boldsymbol{i}}^t + \boldsymbol{\theta}_{\boldsymbol{i}}^{t+1} - \boldsymbol{\rho}^{t+1}, \boldsymbol{i} \le \boldsymbol{N}$$

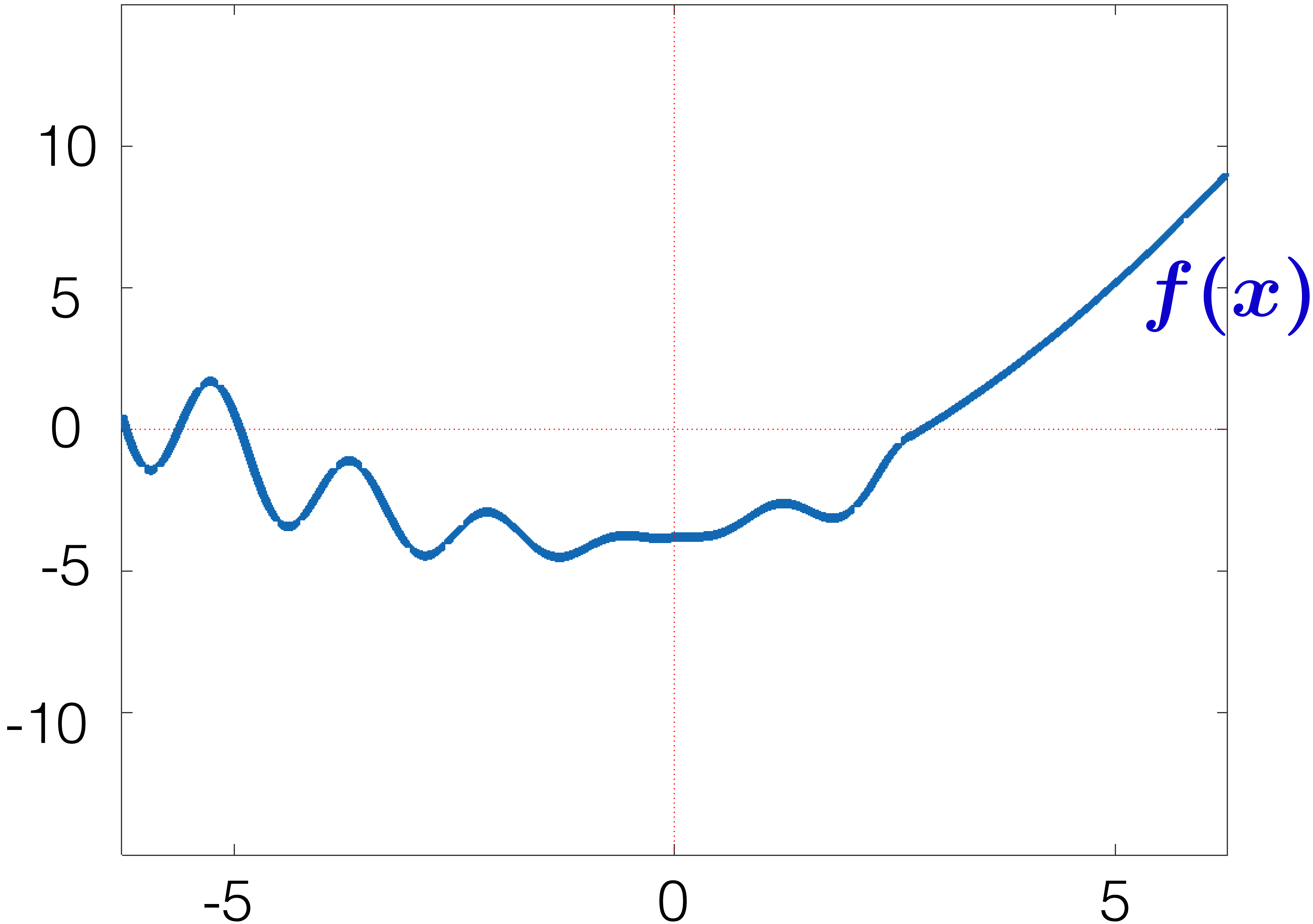
Dual Methods

Def

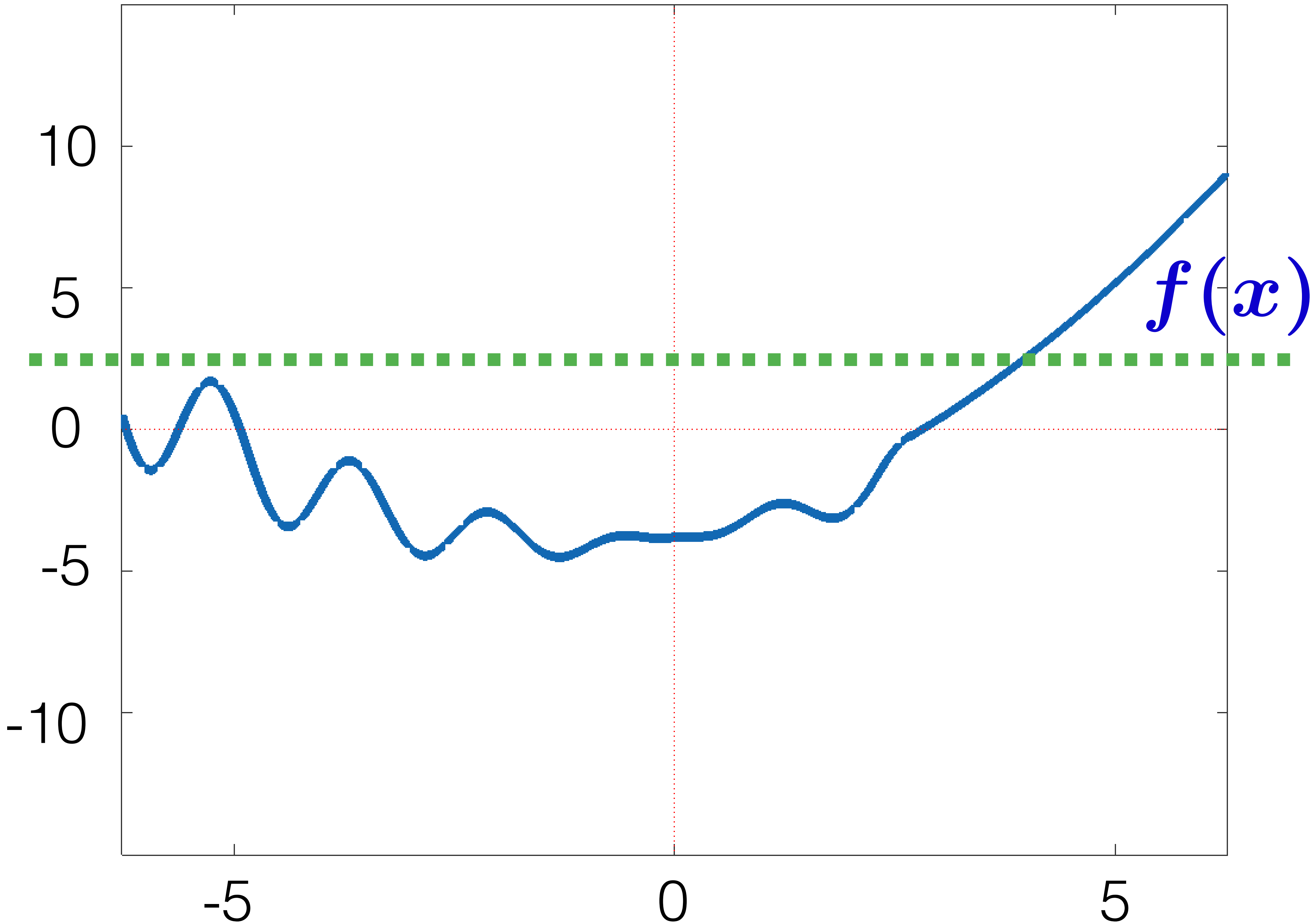
For a (possibly non convex) function $f : \mathbb{R}^p \to \bar{\mathbb{R}}$, the convex conjugate of f is, $\forall \boldsymbol{y} \in \mathbb{R}^p$,

$$f^*(\boldsymbol{y}) = \sup_{\boldsymbol{x} \in \mathbb{R}^p} \langle \boldsymbol{x}, y \rangle - f(\boldsymbol{x})$$

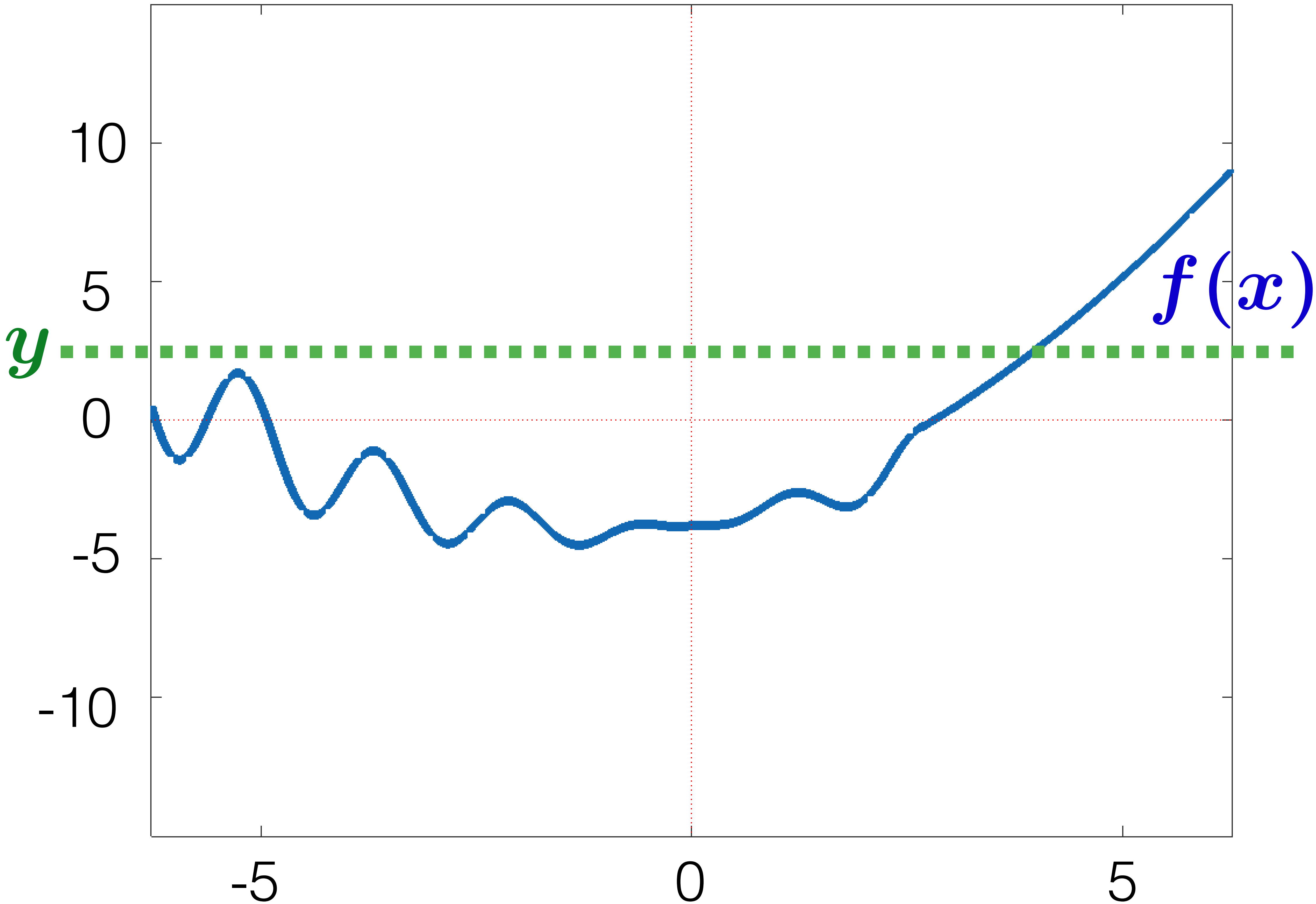
Legendre Transform



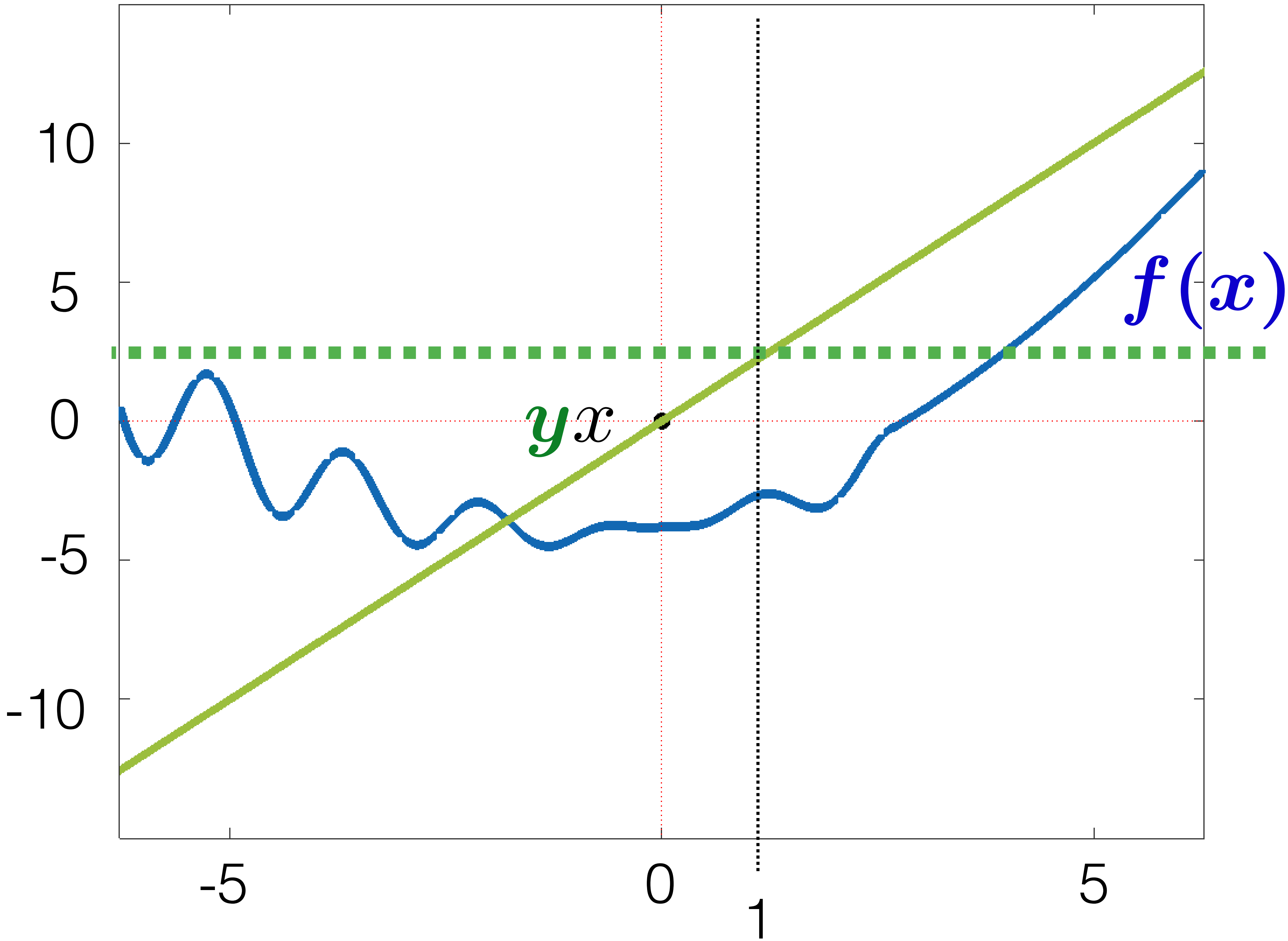
Legendre Transform



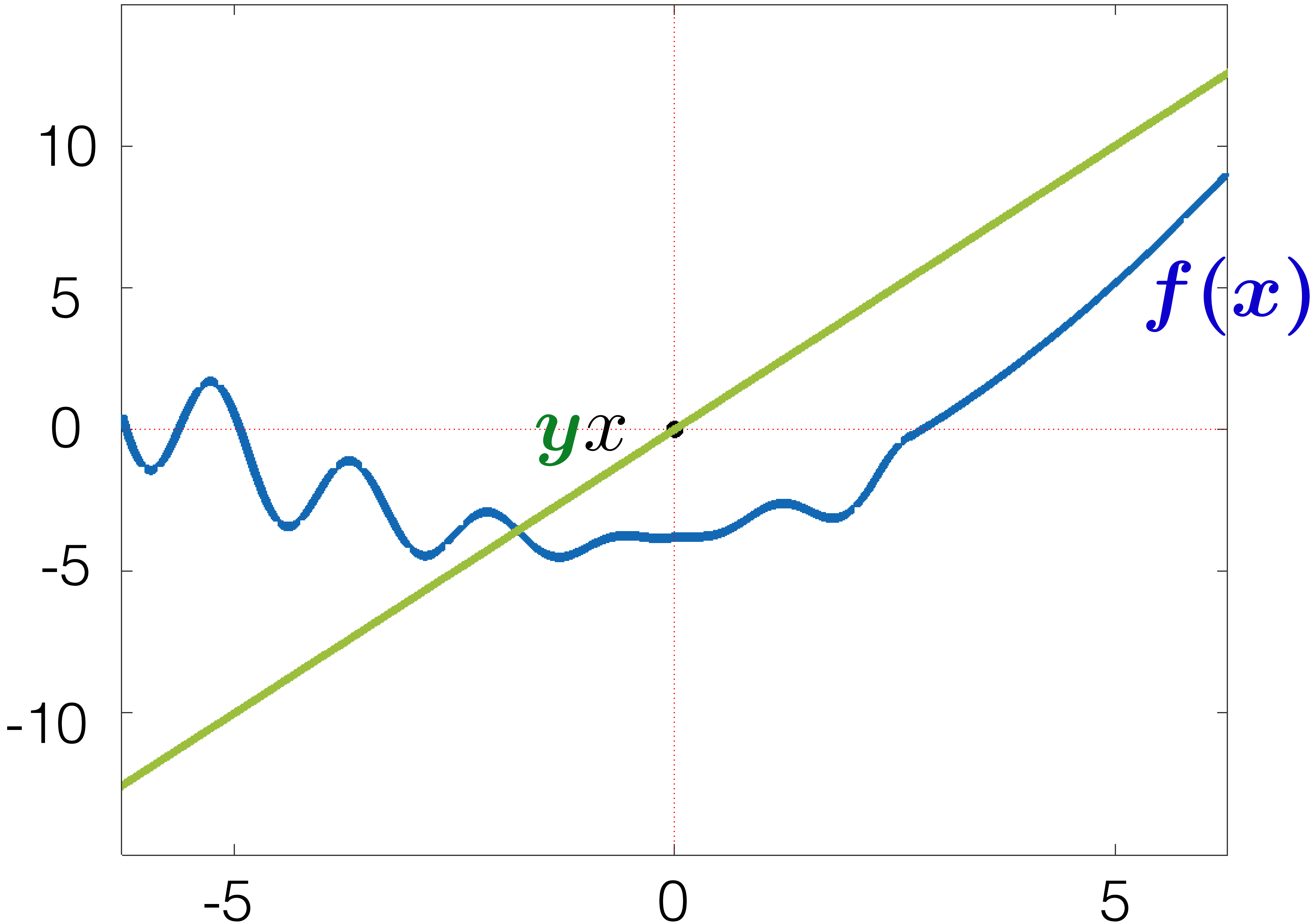
Legendre Transform



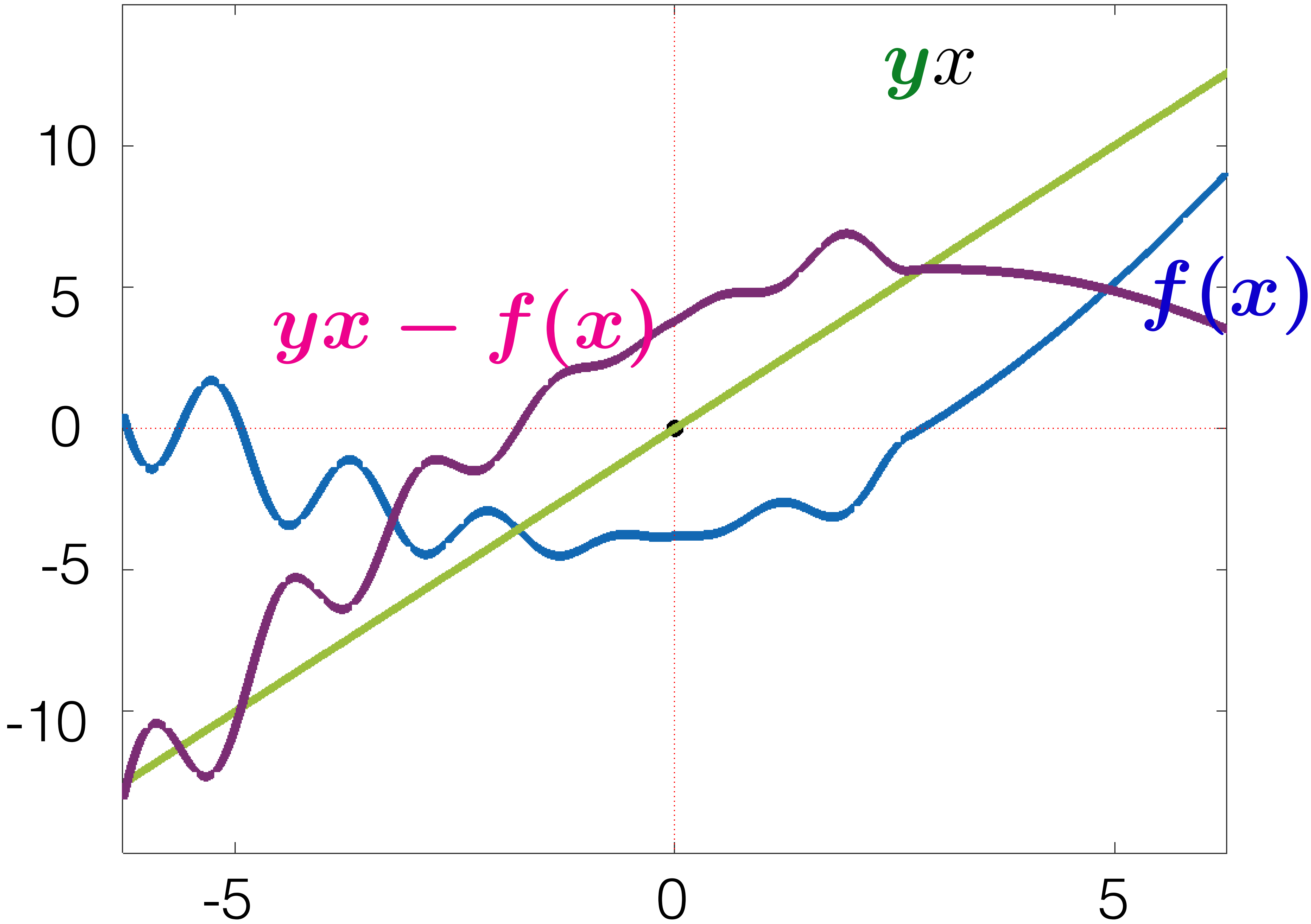
Legendre Transform



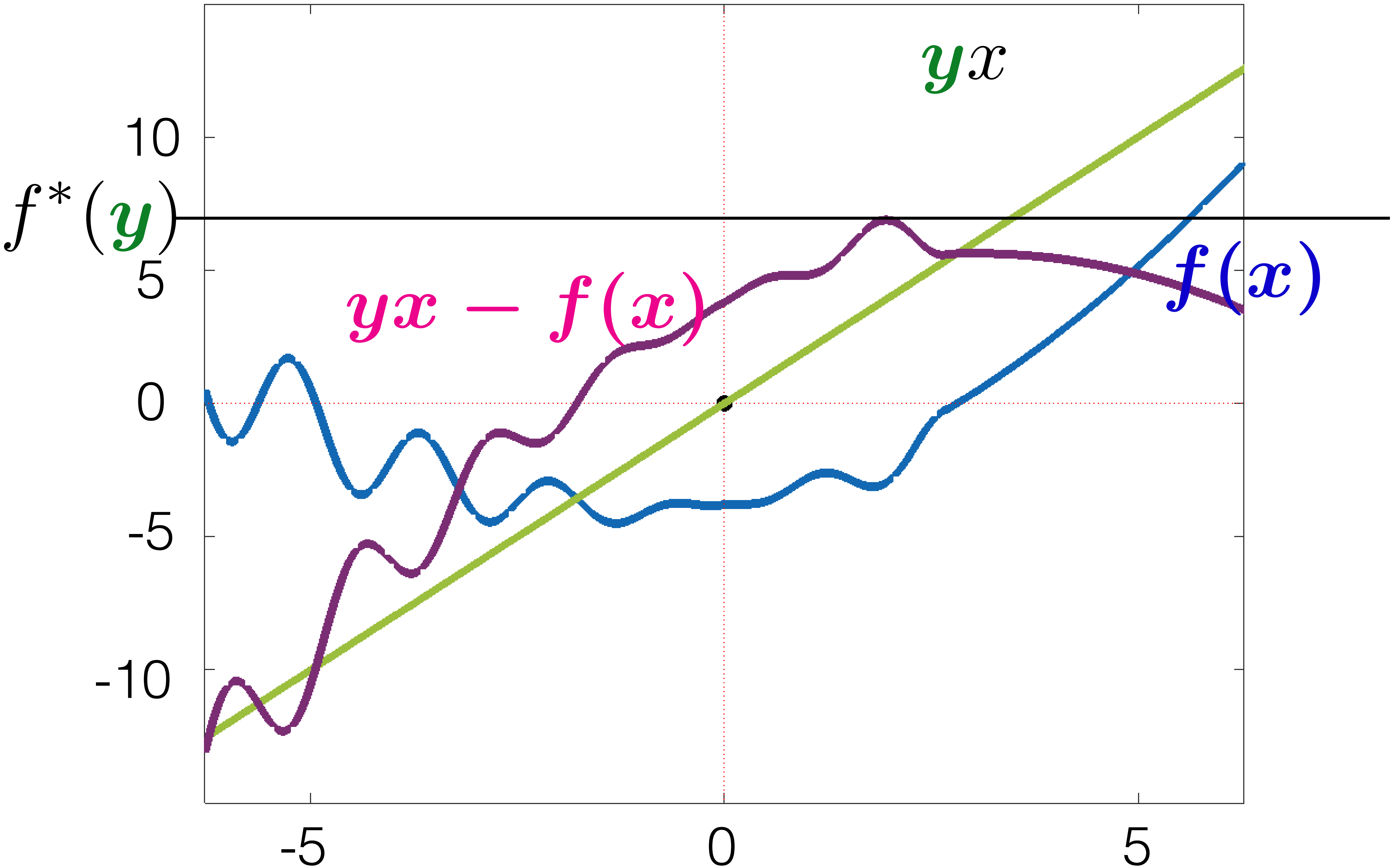
Legendre Transform



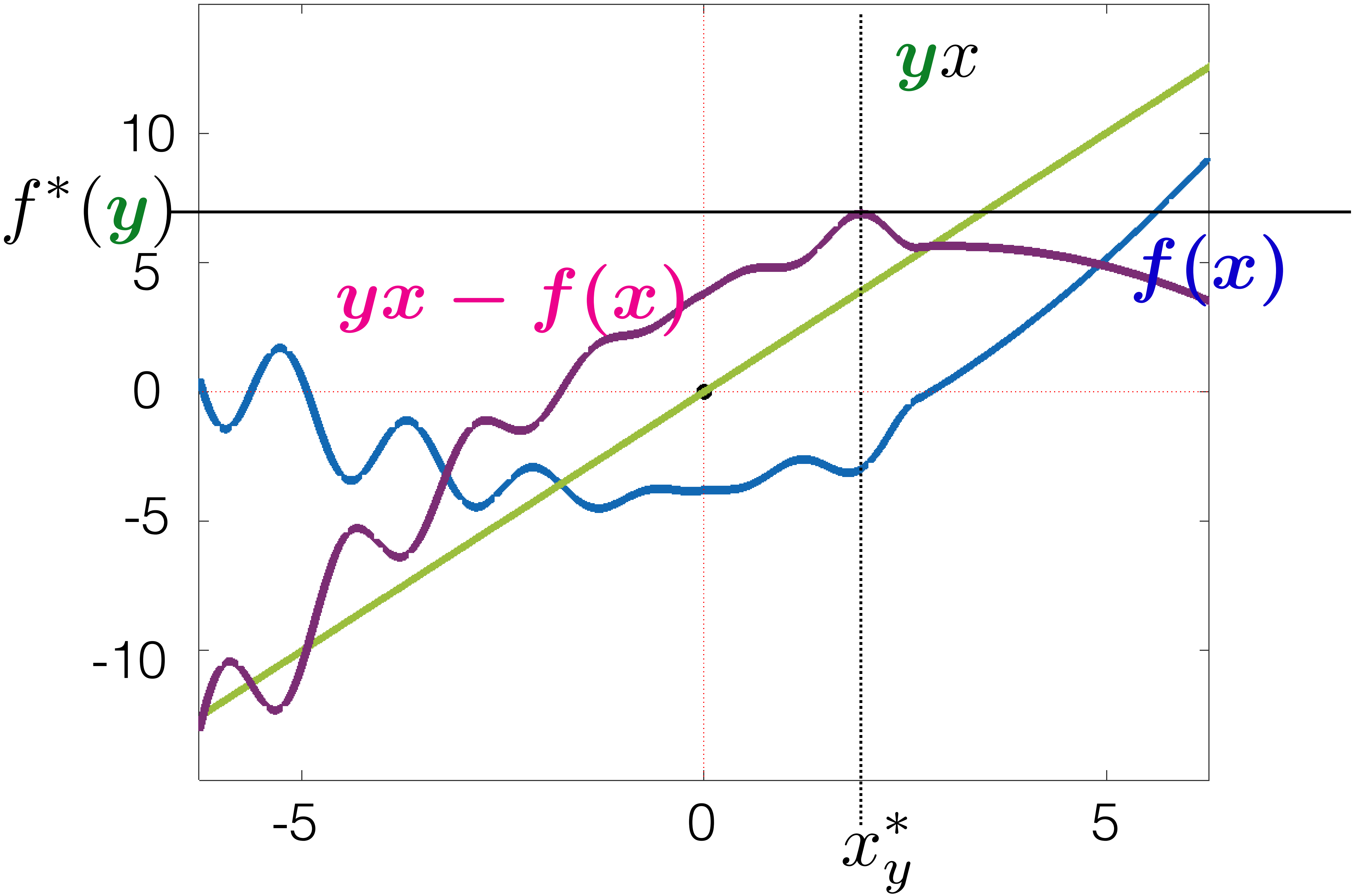
Legendre Transform



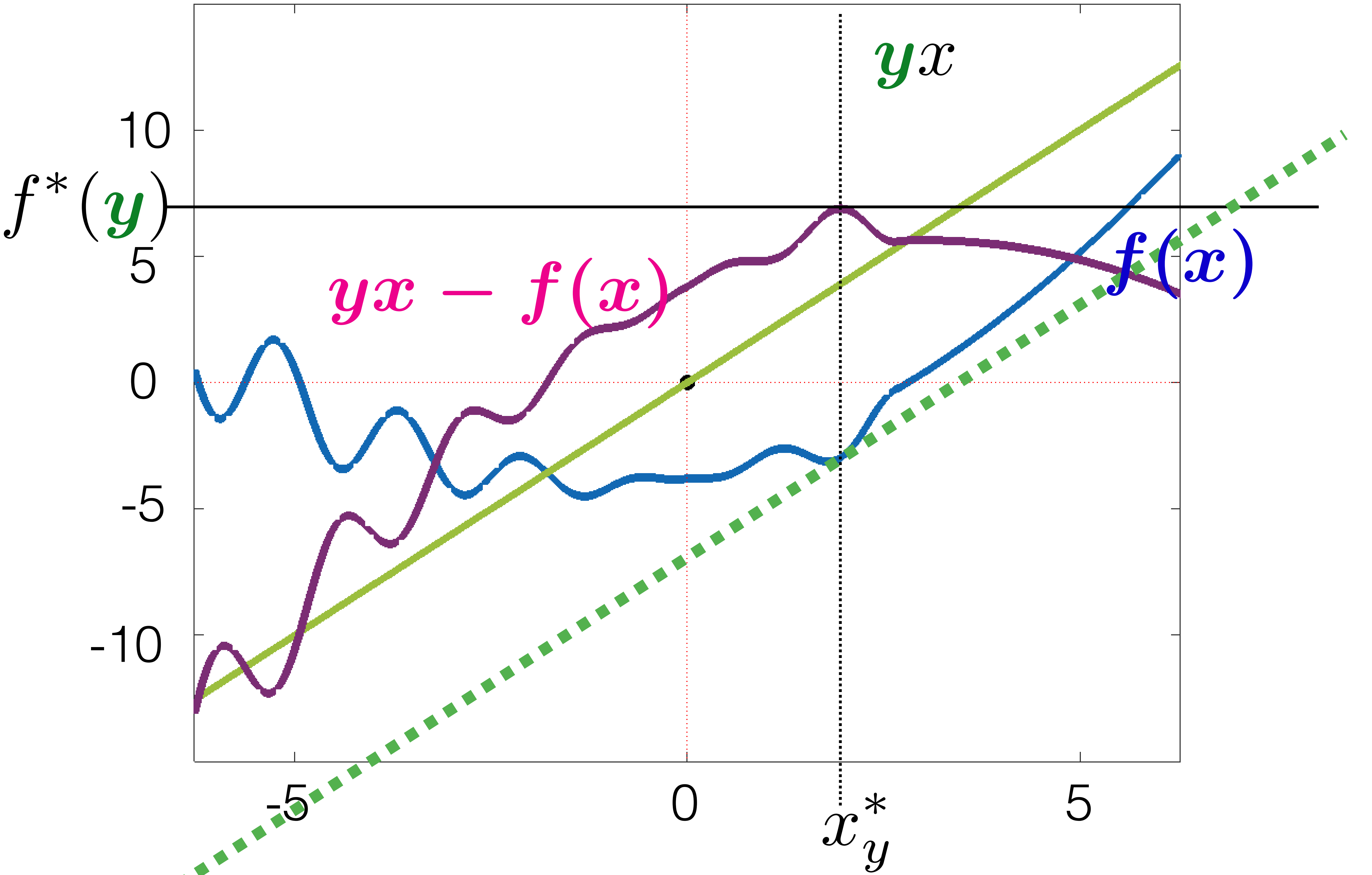
Legendre Transform



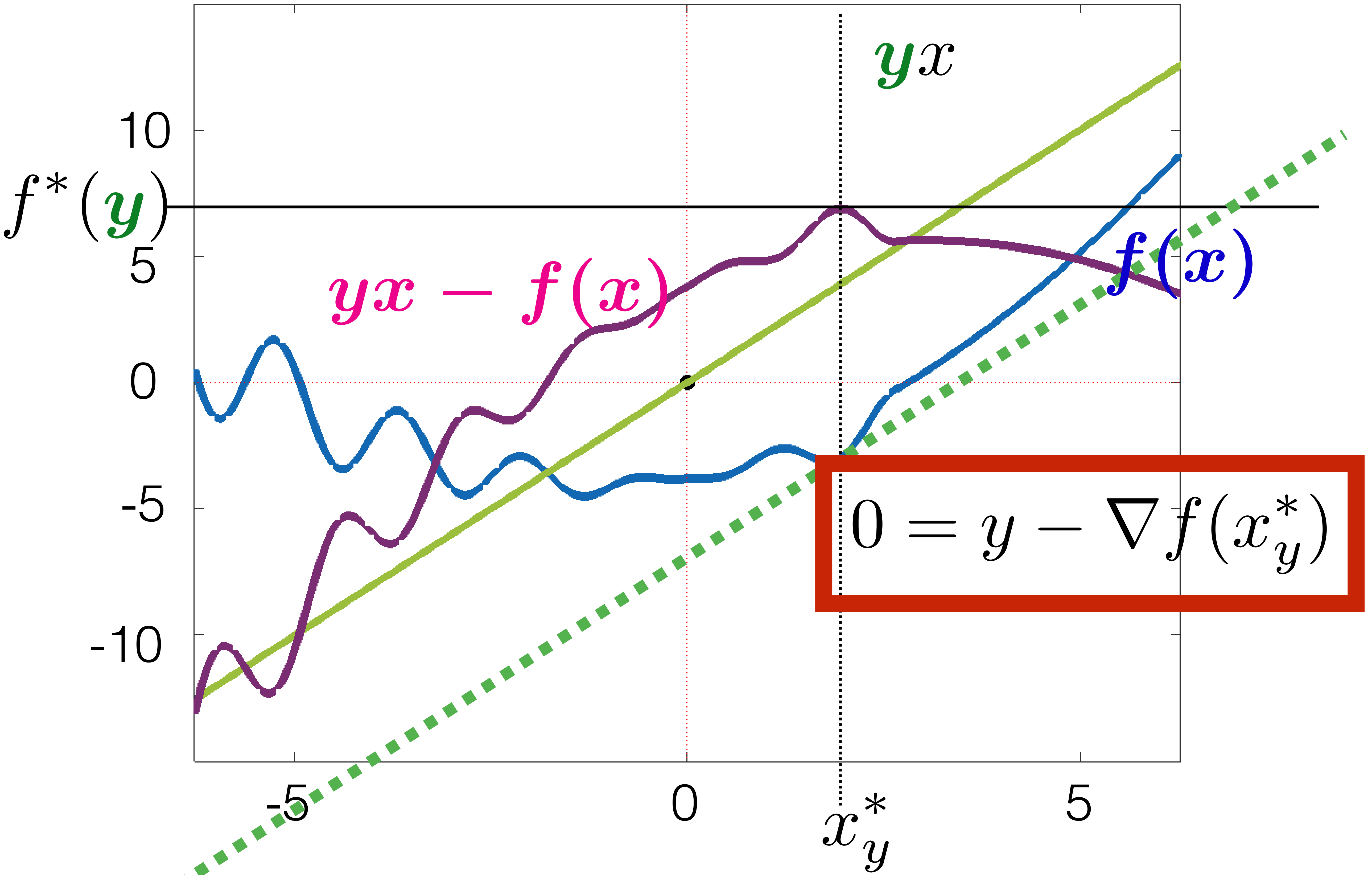
Legendre Transform



Legendre Transform

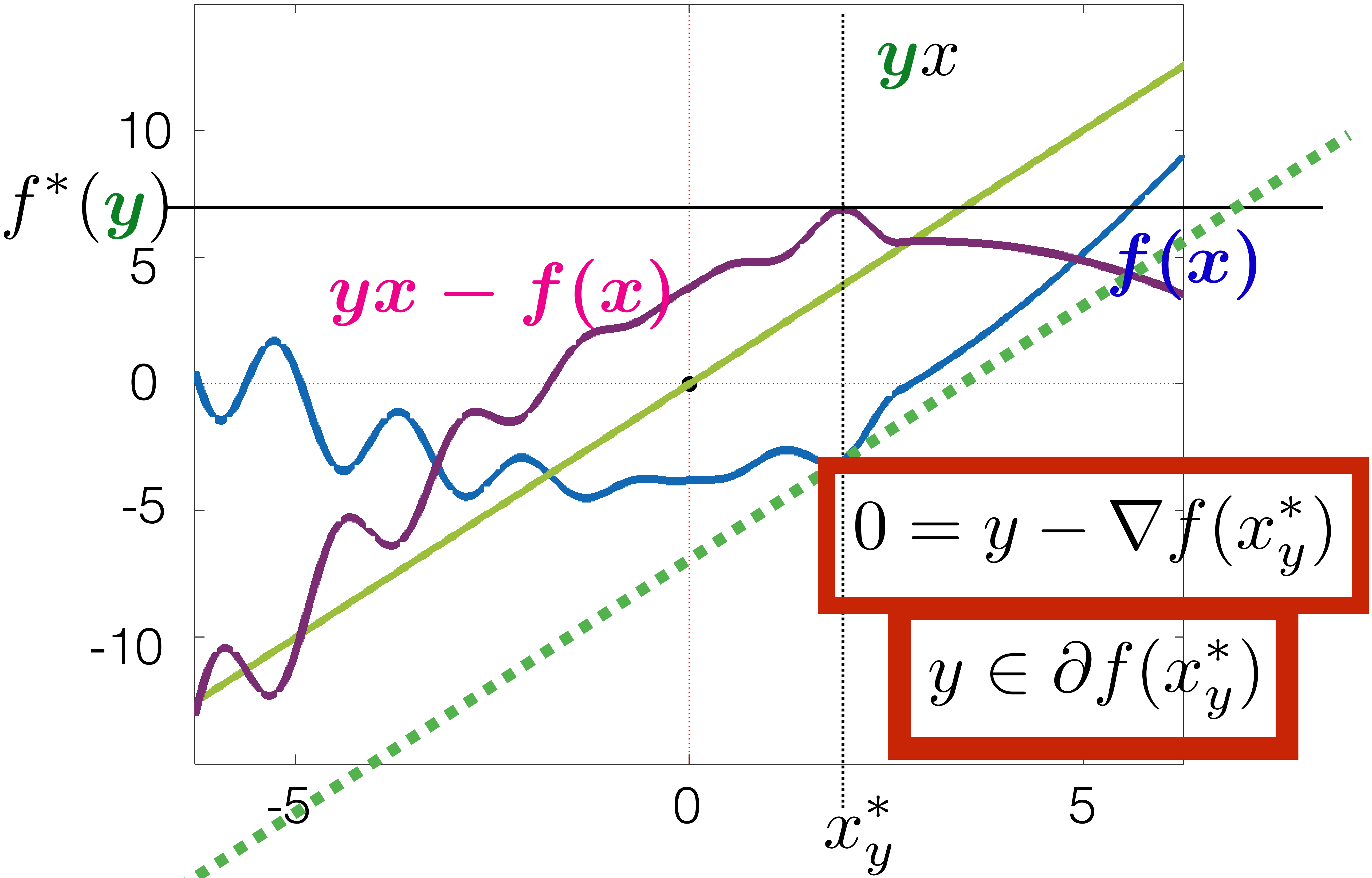


Legendre Transform

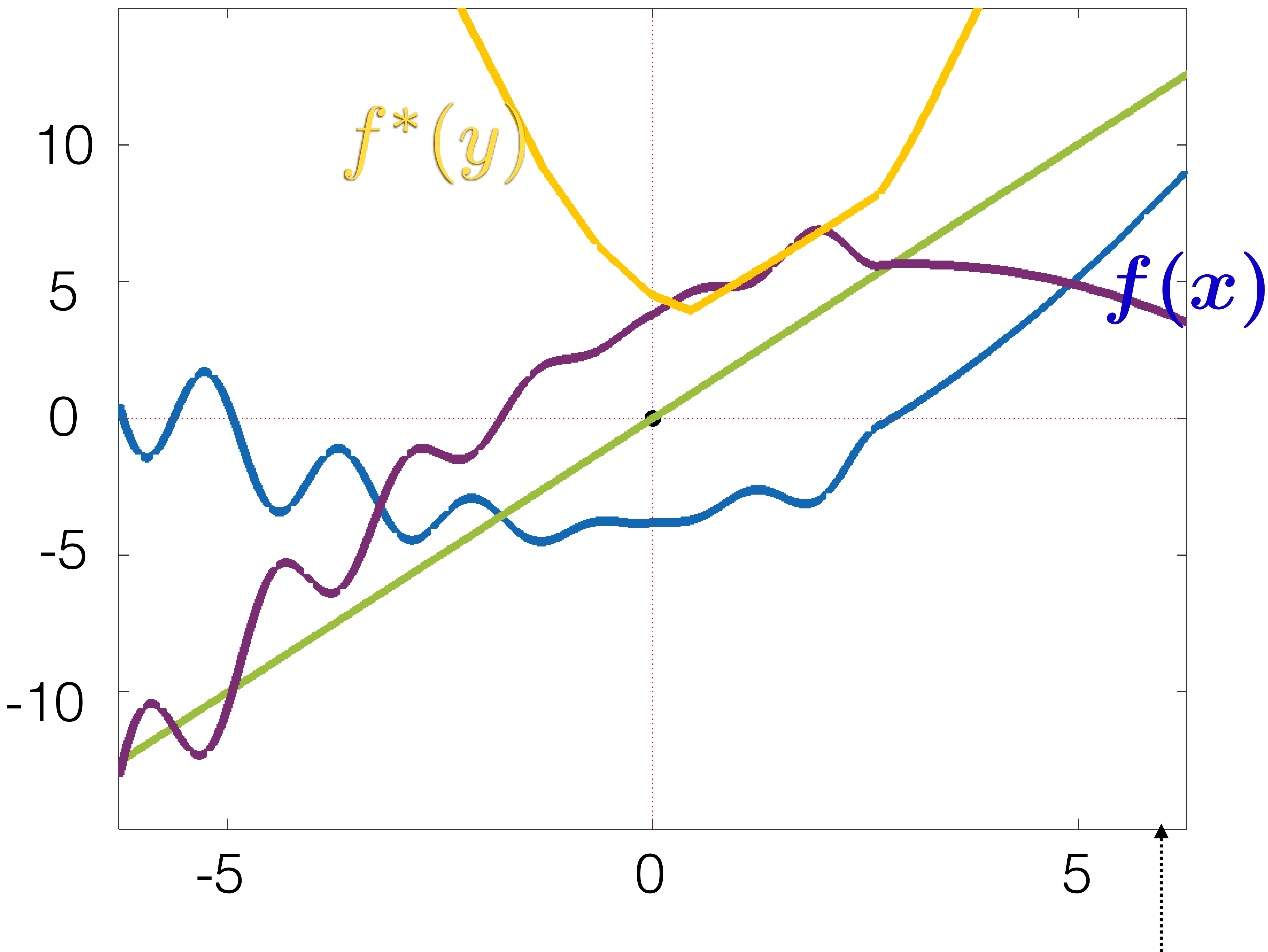


$$\boxed{0 = y - \nabla f(x_y^*)}$$

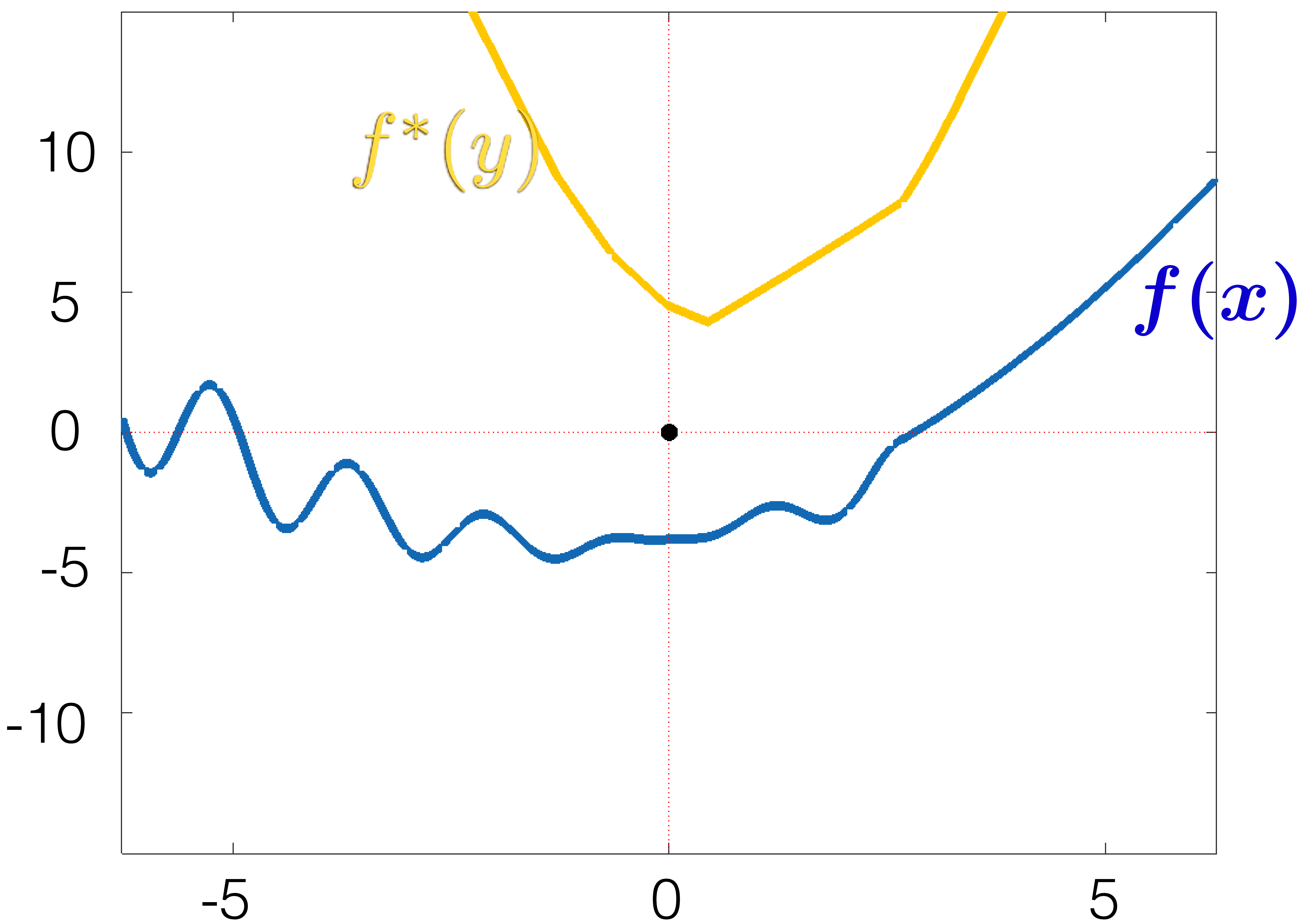
Legendre Transform



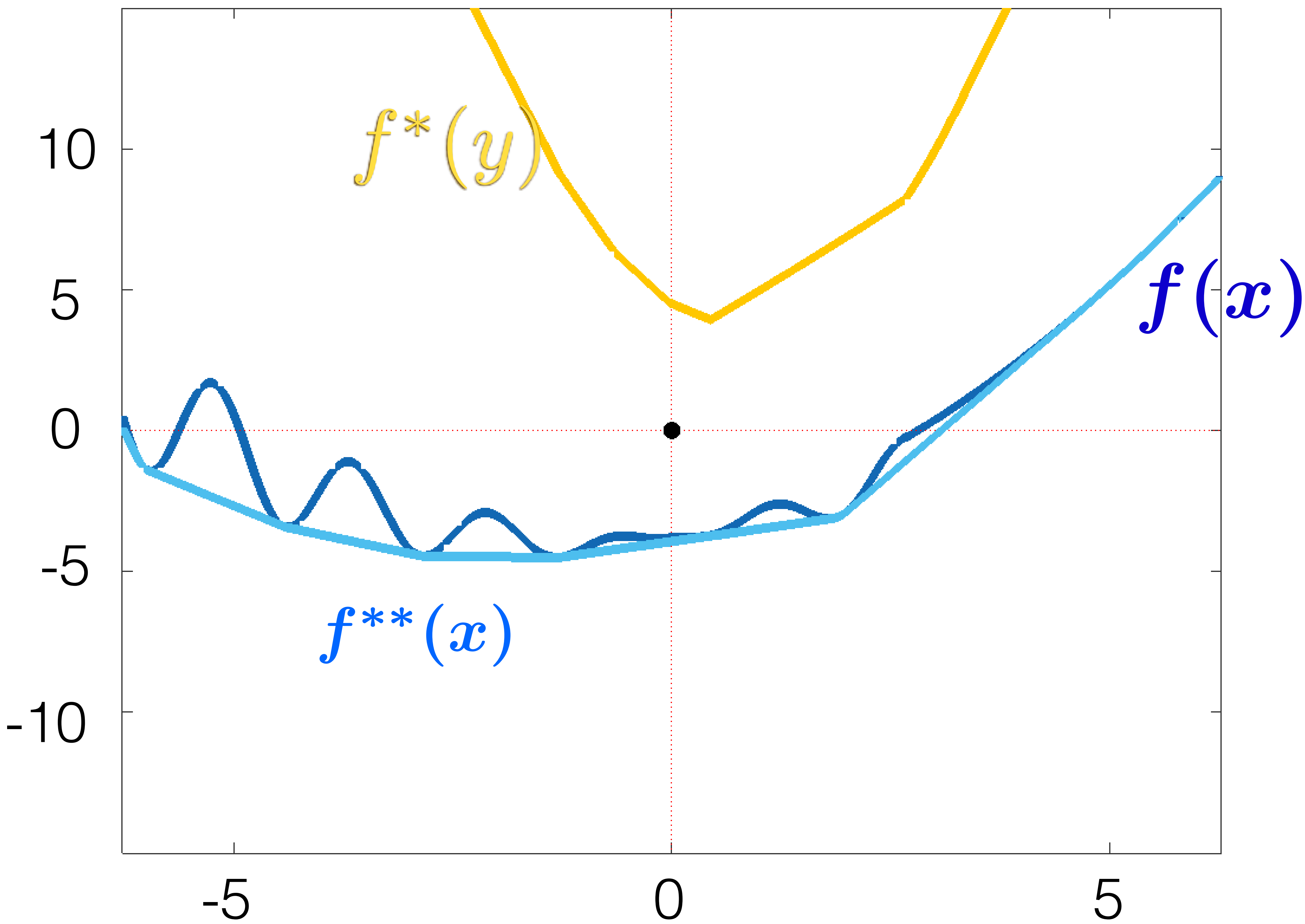
Legendre Transform



Legendre Transform



Legendre Transform



Legendre Transform

Def

For a (possibly non convex) function $f : \mathbb{R}^p \to \bar{\mathbb{R}}$, the convex conjugate of f is $\forall y \in \mathbb{R}^p$,

$$f^*(y) = \sup_{x \in \mathbb{R}^p} \langle x, y \rangle - f(x)$$

	$f(x)$	$f^*(y)$
Squared loss	$\frac{1}{2}x^2$	$\frac{1}{2}y^2$
Hinge loss	$\max\{1-x, 0\}$	$\begin{cases} y & (-1 \leq y \leq 0), \\ \infty & (\text{otherwise}). \end{cases}$
Logistic loss	$\log(1+\exp(-x))$	$\begin{cases} (-y)\log(-y) + (1+y)\log(1+y) & (-1 \leq y \leq 0), \\ \infty & (\text{otherwise}). \end{cases}$
L_1 regularization	$\|x\|_1$	$\begin{cases} 0 & (\max_j \|y_j\| \leq 1), \\ \infty & (\text{otherwise}). \end{cases}$
L_p regularization $(p > 1)$	$\sum_{j=1}^d \|x_j\|^p$	$\sum_{j=1}^d \frac{p-1}{p^{\frac{p}{p-1}}} \|y_j\|^{\frac{p}{p-1}}$

Legendre Transform

Def

For a (possibly non convex) function $f : \mathbb{R}^p \to \bar{\mathbb{R}}$, the convex conjugate of f is $\forall y \in \mathbb{R}^p$,

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f^* is convex, even if f is not.

$$y \in \partial f(x) \Leftrightarrow f(x) + f^*(y) = \langle x, y \rangle \Leftrightarrow x \in \partial f^*(y)$$

$$\forall x, y, f(x) + f^*(y) \geq \langle x, y \rangle$$

Fenchel Duality Theorem

Theorem

Let $f : \mathbb{R}^p \to \bar{R}$ and $g : \mathbb{R}^q \to \bar{R}$ be closed convex, and $A \in \mathbb{R}^{q \times p}$ a linear map. Suppose that either condition (a) or (b) is satisfied. Then

$$\inf_{x \in \mathbb{R}^p} f(x) + g(Ax) = \sup_{y \in \mathbb{R}^q} -f^*(A^T y) - g^*(-y)$$

$(a) \exists x \in \mathbb{R}^p$ s.t. $x \in \text{ri}(\text{dom}(f))$ and $Ax \in \text{ri}(\text{dom}(g))$

$(b) \exists y \in \mathbb{R}^q$ s.t. $A^T y \in \text{ri}(\text{dom}(f^*))$ and $-y \in \text{ri}(\text{dom}(g^*))$

Fenchel Duality and ERM

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} l_{\boldsymbol{\theta}}(z_i) + \psi(\boldsymbol{\theta})$$

$$l_{\boldsymbol{\theta}}(z_i) = l(y_i, x_i^T \boldsymbol{\theta})$$

$$\frac{1}{n} \sum_i l_{\boldsymbol{\theta}}(z_i) = \mathbf{l}(\mathbf{y}, X\boldsymbol{\theta}) = g(X\boldsymbol{\theta})$$

$$X \in \mathbb{R}^{n \times p}$$

$$\sup_{\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}} -\psi^*(-X^T y) - g^*(y) = -\inf_{\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}} g^*(y) + \psi^*(-X^T y)$$

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Dual Methods

- Set $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$,
- For $k = 1, \ldots, K$
  - $x_i^{k+1} = \underset{y \in \mathbb{R}}{\arg\min}\ f(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, y, x_{i+1}^{k}, \ldots, x_n^{k})$

source: wikipedia

Dual Methods

Reminders on Coordinate Descent

- Set $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$,
- For $k = 1, \ldots, K$
  - $x_i^{k+1} = \underset{y \in \mathbb{R}}{\arg\min}\, f(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, y, x_{i+1}^k, \ldots, x_n^k)$

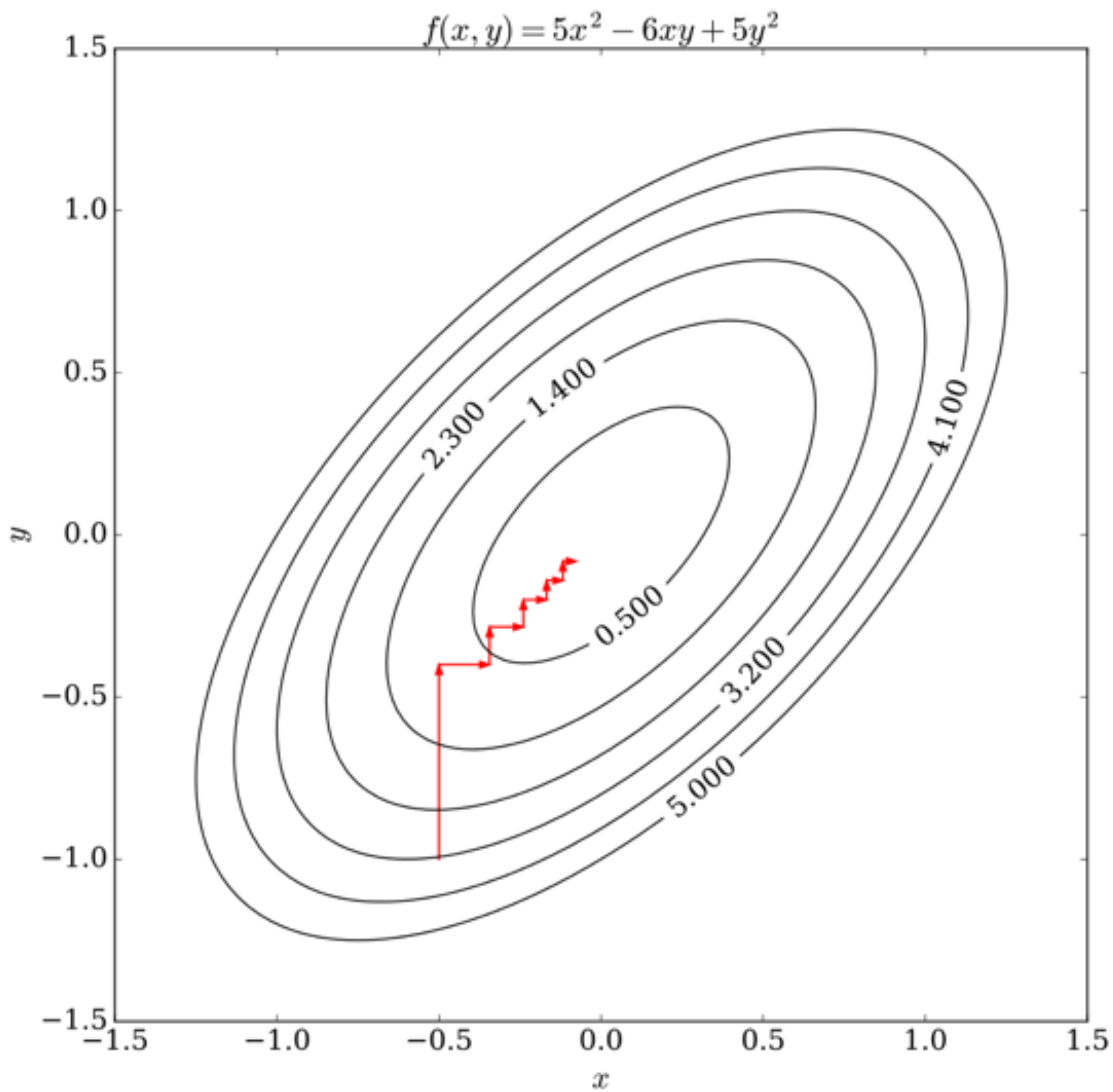
source: wikipedia

Reminders on Coordinate Descent

source: wikipedia

Dual Methods

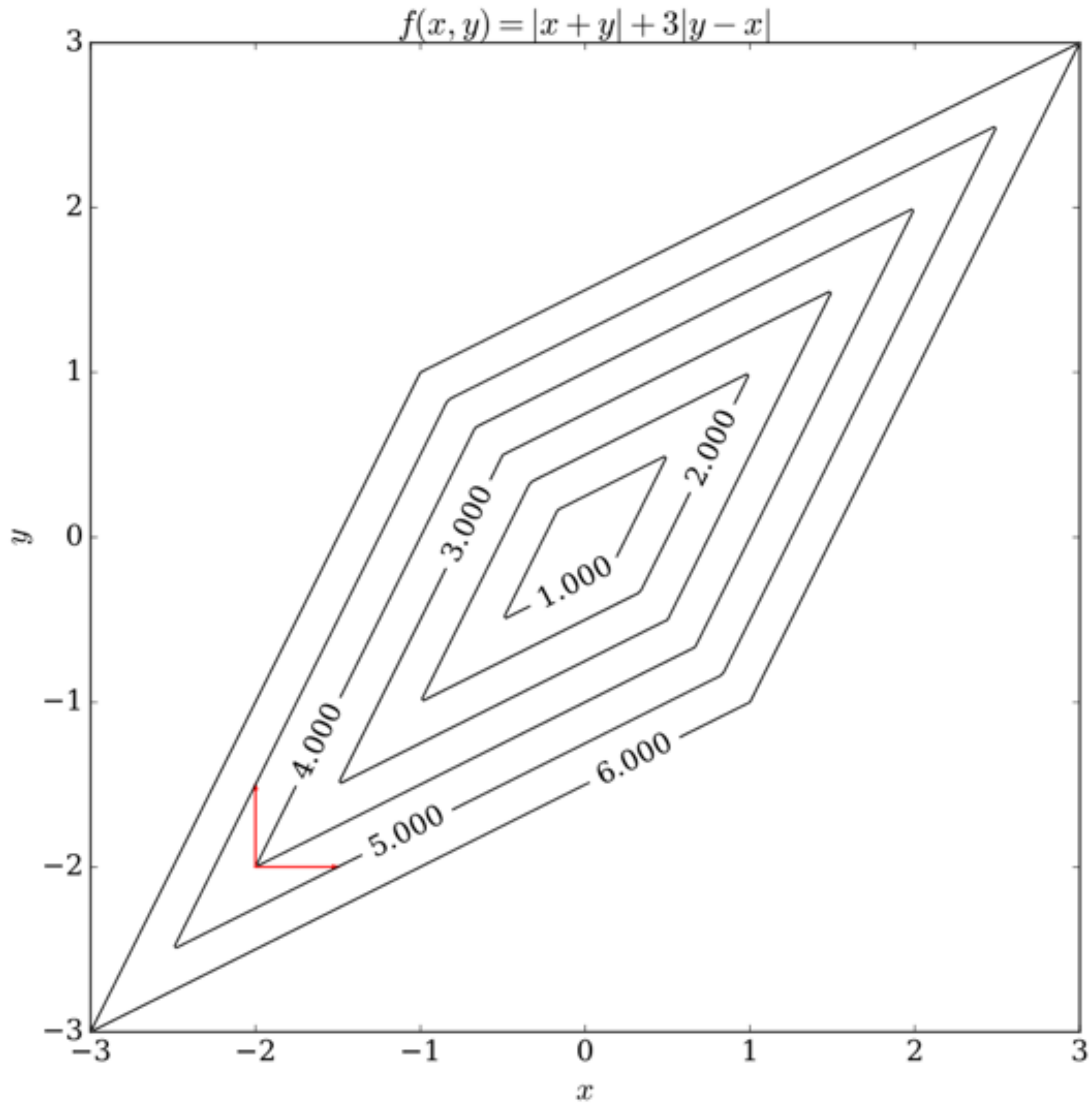
Reminders on Coordinate Descent



source: wikipedia

Dual Methods

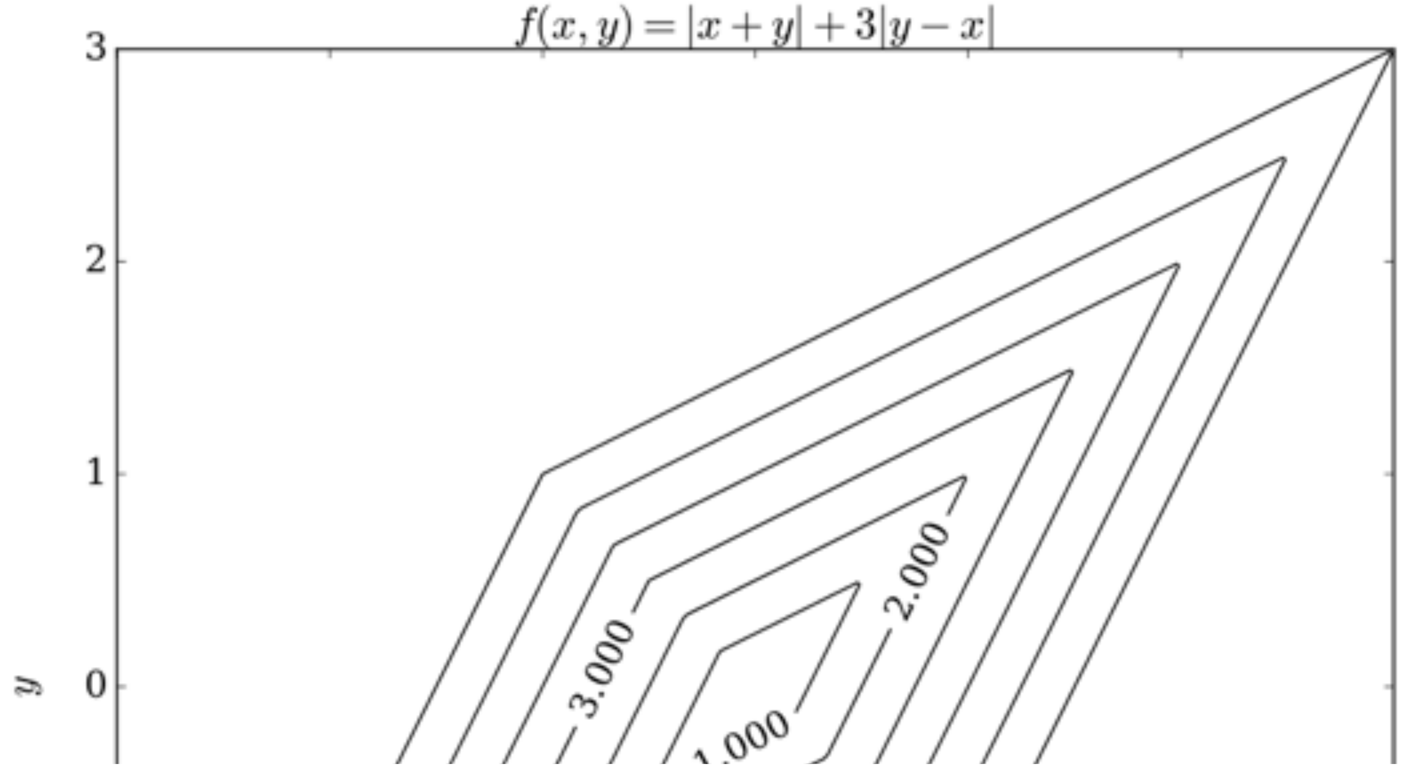
Reminders on Coordinate Descent



source: wikipedia

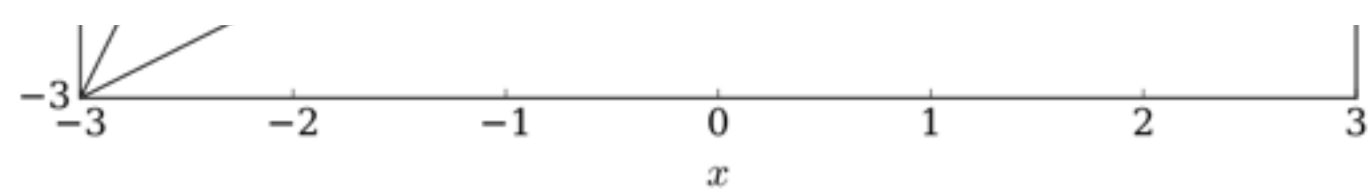
Dual Methods

Reminders on Coordinate Descent



To ensure success of CD, some progress must be guaranteed.

Separability of the objective function helps.



source: wikipedia

Dual Methods

- Set $\theta^0 = (\theta_1^0, \ldots, \theta_p^0)$,
- For $k = 1, \ldots, K$
  - Sample j.
  - Compute $g_j = \partial f(\theta)/\partial\theta_j$
  - $\theta_j \leftarrow \arg\min_{y \in \mathbb{R}} g_j y + \psi_j(y) + \frac{1}{2\eta_t}\|y - \theta_j\|^2$

source: wikipedia

Dual Methods

Coordinate Descent on Primal Problem

- Set $\theta^0 = (\theta_1^0, \ldots, \theta_p^0)$,
- For $k = 1, \ldots, K$
  - Sample j.
  - Compute $g_j = \partial f(\theta) / \partial \theta_j$
  - $\theta_j \leftarrow \arg\min_{y \in \mathbb{R}} g_j y + \psi_j(y) + \frac{1}{2\eta_t} \|y - \theta_j\|^2$

Regularizer must be separable.

source: wikipedia

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Theorem

Let $f : \mathbb{R}^p \to \bar{R}$ and $g : \mathbb{R}^q \to \bar{R}$ be closed convex, and $A \in \mathbb{R}^{q \times p}$ a linear map. Suppose that either condition (a) or (b) is satisfied. Then

$$\inf_{x \in \mathbb{R}^p} f(x) + g(Ax) = \sup_{y \in \mathbb{R}^q} -f^*(A^T y) - g^*(-y)$$

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} l_{\boldsymbol{\theta}}(z_i) + \psi(\boldsymbol{\theta})$$

$$l_{\boldsymbol{\theta}}(z_i) = l(y_i, x_i^T \boldsymbol{\theta})$$

$$\sup_{\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}} \frac{1}{n} \sum_i l_i^*(y_i) + \psi^*(-X^T y/n)$$

$$\boldsymbol{\theta}^* = \nabla \psi^*(-X^T \boldsymbol{y}^* / n)$$

Fenchel Duality and ERM

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$$\sup_{\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}} \sum_i l_i^*(y_i) + \psi^*(-X^T y)$$

SDCA

SDCA (Shalev-Shwartz and Zhang, 2013a)

Iterate the following for $t = 1, 2, \ldots$

1. Pick up an index $i \in \{1, \ldots, n\}$ uniformly at random.
2. Update the i-th coordinate y_i so that the objective function is decreased.

SDCA

SDCA (Shalev-Shwartz and Zhang, 2013a)

Iterate the following for $t = 1, 2, \ldots$

1. Pick up an index $i \in \{1, \ldots, n\}$ uniformly at random.
2. Update the i-th coordinate y_i:
   (let $A_{\setminus i} = [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n]$, and $y_{\setminus i} = (y_j)_{j \neq i}$)

   - $$y_i^{(t)} \in \mathop{\mathrm{argmin}}_{y_i \in \mathbb{R}} \left\{ f_i^*(y_i) + n\psi^*\left(-\frac{1}{n}(a_i y_i + A_{\setminus i} y_{\setminus i}^{(t-1)}) \right) + \frac{1}{2\eta} \|y_i - y_i^{(t-1)}\|^2 \right\},$$
   - $y_j^{(t)} = y_j^{(t-1)}$ (for $j \neq i$).

SDCA

SDCA (linearized version) (Shalev-Shwartz and Zhang, 2013a)

Iterate the following for $t = 1, 2, \ldots$

1. Pick up an index $i \in \{1, \ldots, n\}$ uniformly at random.
2. Calculate $x^{(t-1)} = \nabla \psi^*(-Ay^{(t-1)}/n)$.
3. Update the i-th coordinate y_i:

   - $$y_i^{(t)} \in \operatorname*{argmin}_{y_i \in \mathbb{R}} \left\{ f_i^*(y_i) - \langle x^{(t-1)}, a_i y_i \rangle + \frac{1}{2\eta} \|y_i - y_i^{(t-1)}\|^2 \right\}$$

   - $y_j^{(t)} = y_j^{(t-1)}$ (for $j \neq i$).

SDCA : SVM

$$E(\mathbf{w}) = \frac{\lambda}{2}\|\mathbf{w}\|^2 + \frac{1}{n}\sum_{i=1}^{n} \ell_i(\langle \mathbf{w}, \mathbf{x} \rangle).$$

$$D(\boldsymbol{\alpha}) = -\frac{1}{2\lambda n^2}\boldsymbol{\alpha}^\top X^\top X \boldsymbol{\alpha} + \frac{1}{n}\sum_{i=1}^{n} -\ell_i^*(-\alpha_i)$$

http://www.vlfeat.org/api/svm-sdca.html

SDCA : SVM

Name	Loss $\ell_i(z)$	Conjugate loss $\ell_i^*(u)$
Hinge	$\max\{0, 1 - y_i z\}$	$\ell_i^*(u) = \begin{cases} y_i u, & -1 \leq y_i u \leq 0, \\ +\infty, & \text{otherwise} \end{cases}$
Square hinge	$\max\{0, 1 - y_i z\}^2$	$\ell_i^*(u) = \begin{cases} y_i u + \frac{u^2}{4}, & y_i u \leq 0, \\ +\infty, & \text{otherwise} \end{cases}$
Linear or l1	$\lvert y_i - z \rvert$	$\ell_i^*(u) = \begin{cases} y_i u, & -1 \leq y_i u \leq 1, \\ +\infty, & \text{otherwise} \end{cases}$
Square or l2	$(y_i - z)^2$	$\ell_i^*(u) = y_i u + \frac{u^2}{4}$
Insensitive l1	$\max\{0, \lvert y_i - z \rvert - \epsilon\}.$	
Logistic	$\log(1 + e^{-y_i z})$	$\ell_i^*(u) = \begin{cases} (1+u)\log(1+u) - u\log(-u), & -1 \leq y_i u \leq 0, \\ +\infty, & \text{otherwise} \end{cases}$

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SDCA : SVM

$$E(\mathbf{w}) = \frac{\lambda}{2}\|\mathbf{w}\|^2 + \frac{1}{n}\sum_{i=1}^{n} \ell_i(\langle \mathbf{w}, \mathbf{x}\rangle).$$

$$D(\boldsymbol{\alpha}) = -\frac{1}{2\lambda n^2}\boldsymbol{\alpha}^\top X^\top X\boldsymbol{\alpha} + \frac{1}{n}\sum_{i=1}^{n} -\ell_i^*(-\alpha_i)$$

$$\mathbf{w}(\boldsymbol{\alpha}) = \frac{1}{\lambda n}\sum_{i=1}^{n} \mathbf{x}_i\alpha_i = \frac{1}{\lambda n}X\boldsymbol{\alpha}.$$

http://www.vlfeat.org/api/svm-sdca.html

SDCA, SVM ascent

$$D(\boldsymbol{\alpha}_t + \mathbf{e}_q \Delta\alpha_q) = \text{const.} - \frac{1}{2\lambda n^2}\mathbf{x}_q^\top \mathbf{x}_q (\Delta\alpha_q)^2 - \frac{1}{n}\mathbf{x}_q^\top \frac{X\alpha_t}{\lambda n}\Delta\alpha_q - \frac{1}{n}\ell_q^*(-\alpha_q - \Delta\alpha_q)$$

$$\mathbf{w}_t = \frac{X\boldsymbol{\alpha}_t}{\lambda n}, \qquad \mathbf{w}_{t+1} = \mathbf{w}_t + \frac{1}{\lambda n}\mathbf{x}_q \mathbf{e}_q \Delta\alpha_q.$$

http://www.vlfeat.org/api/svm-sdca.html

SDCA, SVM ascent

$$D(\boldsymbol{\alpha}_t + \mathbf{e}_q \Delta\alpha_q) = \text{const.} - \frac{1}{2\lambda n^2}\mathbf{x}_q^\top \mathbf{x}_q (\Delta\alpha_q)^2 - \frac{1}{n}\mathbf{x}_q^\top \frac{X\alpha_t}{\lambda n}\Delta\alpha_q - \frac{1}{n}\ell_q^*(-\alpha_q - \Delta\alpha_q)$$

$$D(\boldsymbol{\alpha}_t + \mathbf{e}_q \Delta\alpha_q) \propto -\frac{A}{2}(\Delta\alpha_q)^2 - B\Delta\alpha_q - \ell_q^*(-\alpha_q - \Delta\alpha_q),$$

$$A = \frac{1}{\lambda n}\mathbf{x}_q^\top \mathbf{x}_q = \frac{1}{\lambda n}\|\mathbf{x}_q\|^2,$$

$$B = \mathbf{x}_q^\top \frac{X\boldsymbol{\alpha}_t}{\lambda n} = \mathbf{x}_q^\top \mathbf{w}_t.$$

$$\mathbf{w}_t = \frac{X\boldsymbol{\alpha}_t}{\lambda n}, \qquad \mathbf{w}_{t+1} = \mathbf{w}_t + \frac{1}{\lambda n}\mathbf{x}_q \mathbf{e}_q \Delta\alpha_q.$$

http://www.vlfeat.org/api/svm-sdca.html

SDCA, SVM ascent, hinge

$$\ell_q^*(u) = \begin{cases} y_q u, & -1 \leq y_q u \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Setting derivative to 0

$$\Delta\alpha_q = y_q \max\{0, \min\{1, y_q(\widetilde{\Delta\alpha_q} + \alpha_q)\}\} - \alpha_q.$$

http://www.vlfeat.org/api/svm-sdca.html

COCOA: A Dual Approach

$$\{1, \dots, n\} = \bigcup_{k=1}^{K} G_k, \;\; G_k \cap G_{k'} = \emptyset.$$

$$\begin{aligned} D(y) &= \frac{1}{n} \sum_{i=1}^{n} f_i^*(y_i) + \psi^* \left(-\frac{1}{n} A y \right) \\ &= \frac{1}{n} \sum_{k=1}^{K} \underbrace{\left(\sum_{i \in G_k} f_i^*(y_i) \right)}_{\text{Divided into } K \text{ groups}} + \underbrace{\psi^* \left(- \frac{1}{n} \sum_{k=1}^{K} A_{G_k} y_{G_k} \right)}_{\text{needs synchronization}} \end{aligned}$$

source: T. Suzuki

COCOA: A Dual Approach

Samples divided into subsets $\{1,\dots,n\} = \bigcup_{k=1}^{K} G_k,\ \ G_k \cap G_{k'} = \emptyset.$

$$
\begin{aligned}
D(y) &= \frac{1}{n}\sum_{i=1}^{n} f_i^*(y_i) + \psi^*\left(-\frac{1}{n}Ay\right) \\
&= \frac{1}{n}\sum_{k=1}^{K} \underbrace{\left(\sum_{i\in G_k} f_i^*(y_i)\right)}_{\text{Divided into } K \text{ groups}} + \underbrace{\psi^*\left(-\frac{1}{n}\sum_{k=1}^{K} A_{G_k} y_{G_k}\right)}_{\text{needs synchronization}}
\end{aligned}
$$

source: T. Suzuki

COCOA: Ex. Quadratic Reg.

$$\min_{w \in \mathbb{R}^d} \quad \left[P(\boldsymbol{w}) := \frac{\lambda}{2} \|\boldsymbol{w}\|^2 + \frac{1}{n} \sum_{i=1}^{n} \ell_i(\boldsymbol{w}^T \boldsymbol{x}_i) \right]$$

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^n} \quad \left[D(\boldsymbol{\alpha}) := -\frac{\lambda}{2} \|A\boldsymbol{\alpha}\|^2 - \frac{1}{n} \sum_{i=1}^{n} \ell_i^*(-\alpha_i) \right]$$

Algorithm 1: COCOA: Communication-Efficient Distributed Dual Coordinate Ascent

Input: $T \geq 1$, scaling parameter $1 \leq \beta_K \leq K$ (default: $\beta_K := 1$).
Data: $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$ distributed over K machines
Initialize: $\boldsymbol{\alpha}_{[k]}^{(0)} \leftarrow \mathbf{0}$ for all machines k, and $\boldsymbol{w}^{(0)} \leftarrow \mathbf{0}$
for $t = 1, 2, \ldots, T$
  for ***all machines*** $k = 1, 2, \ldots, K$ ***in parallel***
    $(\Delta\boldsymbol{\alpha}_{[k]}, \Delta\boldsymbol{w}_k) \leftarrow$ LocalDualMethod$(\boldsymbol{\alpha}_{[k]}^{(t-1)}, \boldsymbol{w}^{(t-1)})$
    $\boldsymbol{\alpha}_{[k]}^{(t)} \leftarrow \boldsymbol{\alpha}_{[k]}^{(t-1)} + \frac{\beta_K}{K} \Delta\boldsymbol{\alpha}_{[k]}$
  end
  reduce $\boldsymbol{w}^{(t)} \leftarrow \boldsymbol{w}^{(t-1)} + \frac{\beta_K}{K} \sum_{k=1}^{K} \Delta\boldsymbol{w}_k$
end

COCOA: Ex. Quadratic Reg.

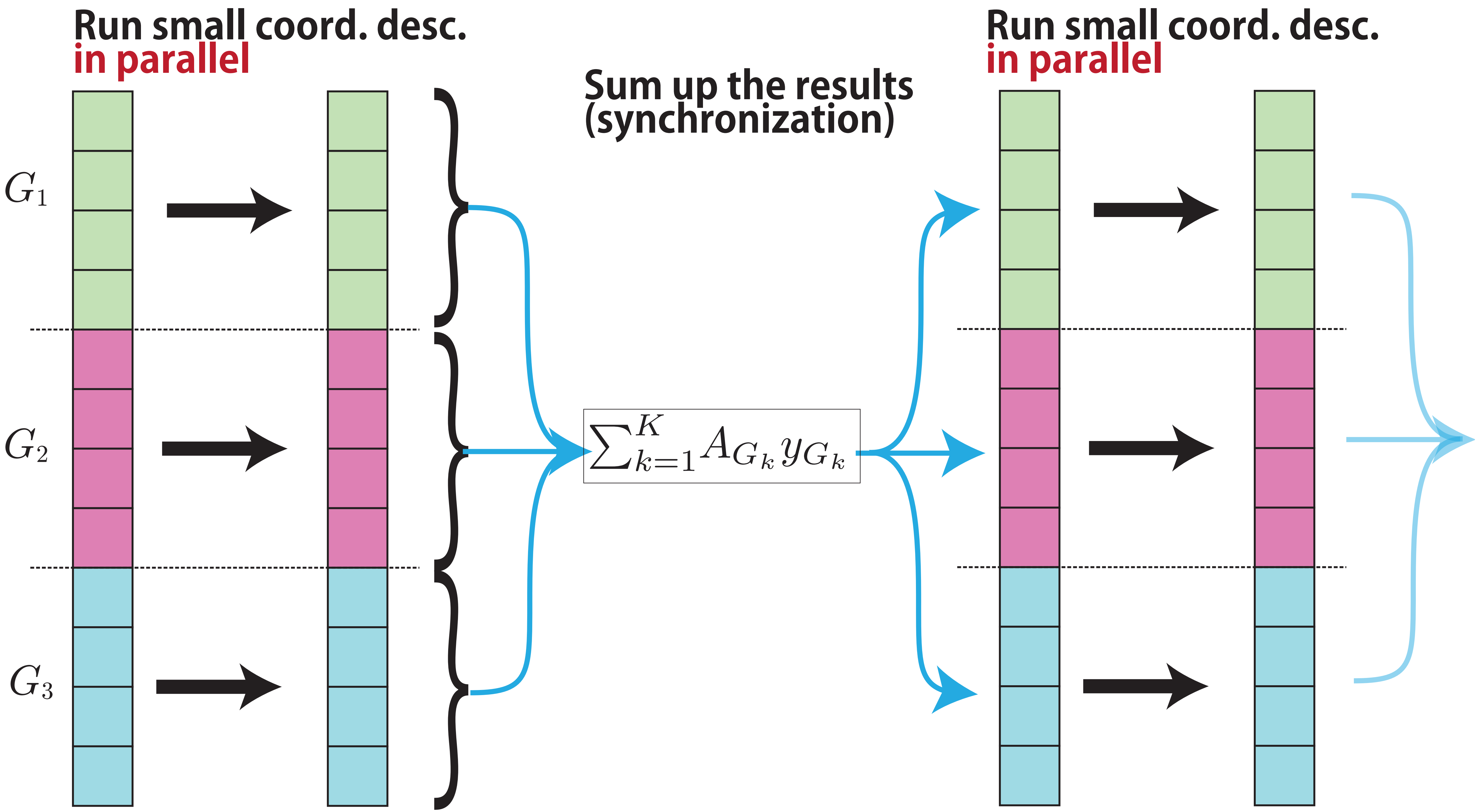
Procedure A: LocalDualMethod: Dual algorithm for prob. (2) on a single coordinate block k

Input: Local $\boldsymbol{\alpha}_{[k]} \in \mathbb{R}^{n_k}$, and $\boldsymbol{w} \in \mathbb{R}^d$ consistent with other coordinate blocks of $\boldsymbol{\alpha}$ s.t. $\boldsymbol{w} = A\boldsymbol{\alpha}$
Data: Local $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^{n_k}$
Output: $\Delta\boldsymbol{\alpha}_{[k]}$ and $\Delta\boldsymbol{w} := A_{[k]}\Delta\boldsymbol{\alpha}_{[k]}$

Procedure B: LocalSDCA: SDCA iterations for problem (2) on a single coordinate block k

Input: $H \geq 1$, $\boldsymbol{\alpha}_{[k]} \in \mathbb{R}^{n_k}$, and $\boldsymbol{w} \in \mathbb{R}^d$ consistent with other coordinate blocks of $\boldsymbol{\alpha}$ s.t. $\boldsymbol{w} = A\boldsymbol{\alpha}$
Data: Local $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^{n_k}$
Initialize: $\boldsymbol{w}^{(0)} \leftarrow \boldsymbol{w}$, $\Delta\boldsymbol{\alpha}_{[k]} \leftarrow \boldsymbol{0} \in \mathbb{R}^{n_k}$
for $h = 1, 2, \ldots, H$
  choose $i \in \{1, 2, \ldots, n_k\}$ *uniformly at random*
  find $\Delta\alpha$ *maximizing* $-\frac{\lambda n}{2}\|\boldsymbol{w}^{(h-1)} + \frac{1}{\lambda n}\Delta\alpha\, \boldsymbol{x}_i\|^2 - \ell_i^*\big(-(\alpha_i^{(h-1)} + \Delta\alpha)\big)$
  $\alpha_i^{(h)} \leftarrow \alpha_i^{(h-1)} + \Delta\alpha$
  $(\Delta\alpha_{[k]})_i \leftarrow (\Delta\alpha_{[k]})_i + \Delta\alpha$
  $\boldsymbol{w}^{(h)} \leftarrow \boldsymbol{w}^{(h-1)} + \frac{1}{\lambda n}\Delta\alpha\, \boldsymbol{x}_i$
end
Output: $\Delta\boldsymbol{\alpha}_{[k]}$ and $\Delta\boldsymbol{w} := A_{[k]}\Delta\boldsymbol{\alpha}_{[k]}$

COCOA


Run small coord. desc.
in parallel
Sum up the results
(synchronization)
Run small coord. desc.
in parallel
G_1
G_2
G_3
$\sum_{k=1}^{K} A_{G_k} y_{G_k}$