Statistical machine learning and convex optimization

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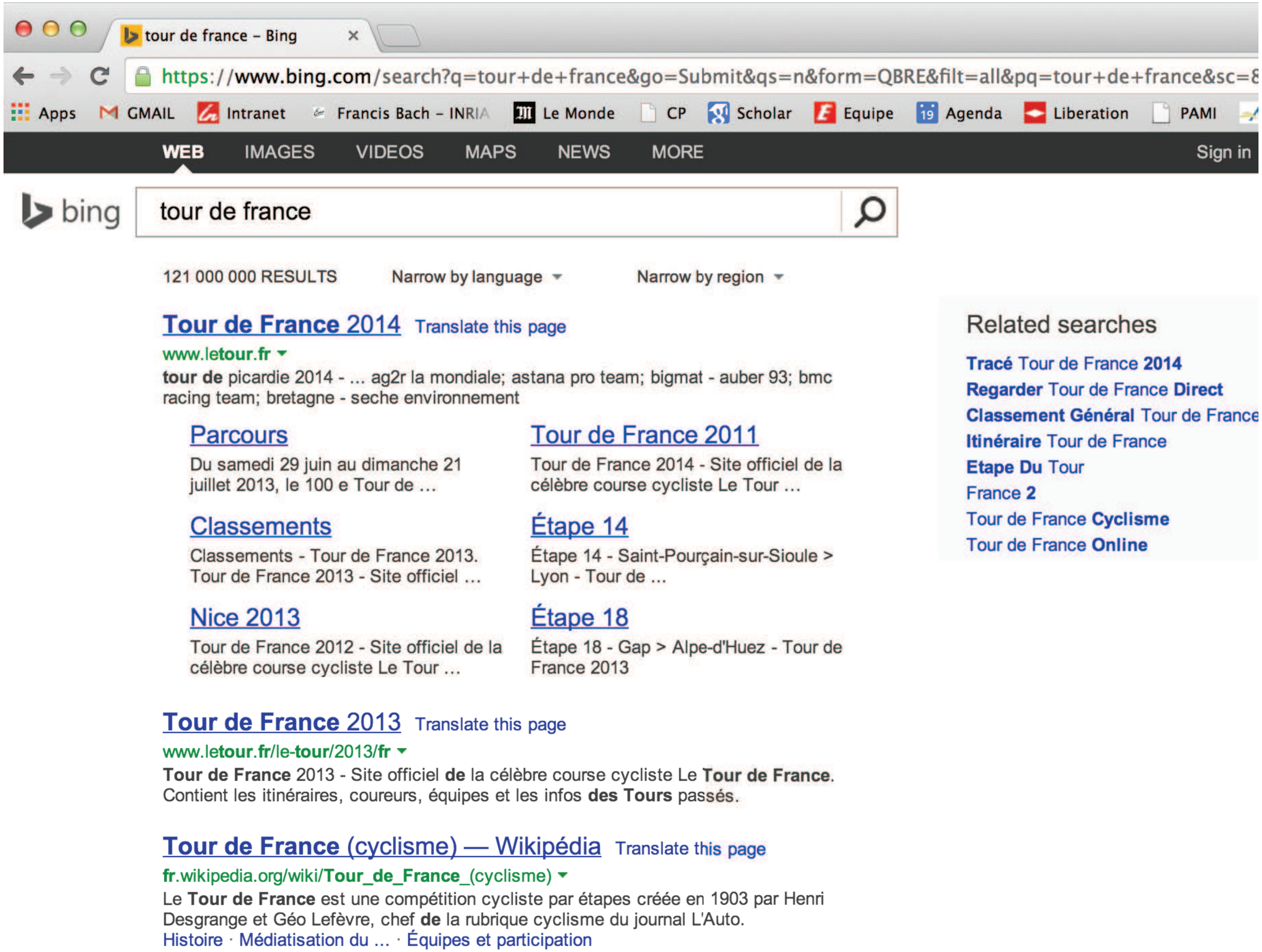
Slides available: `www.di.ens.fr/~fbach/mines_2017_slides_bach.pdf`

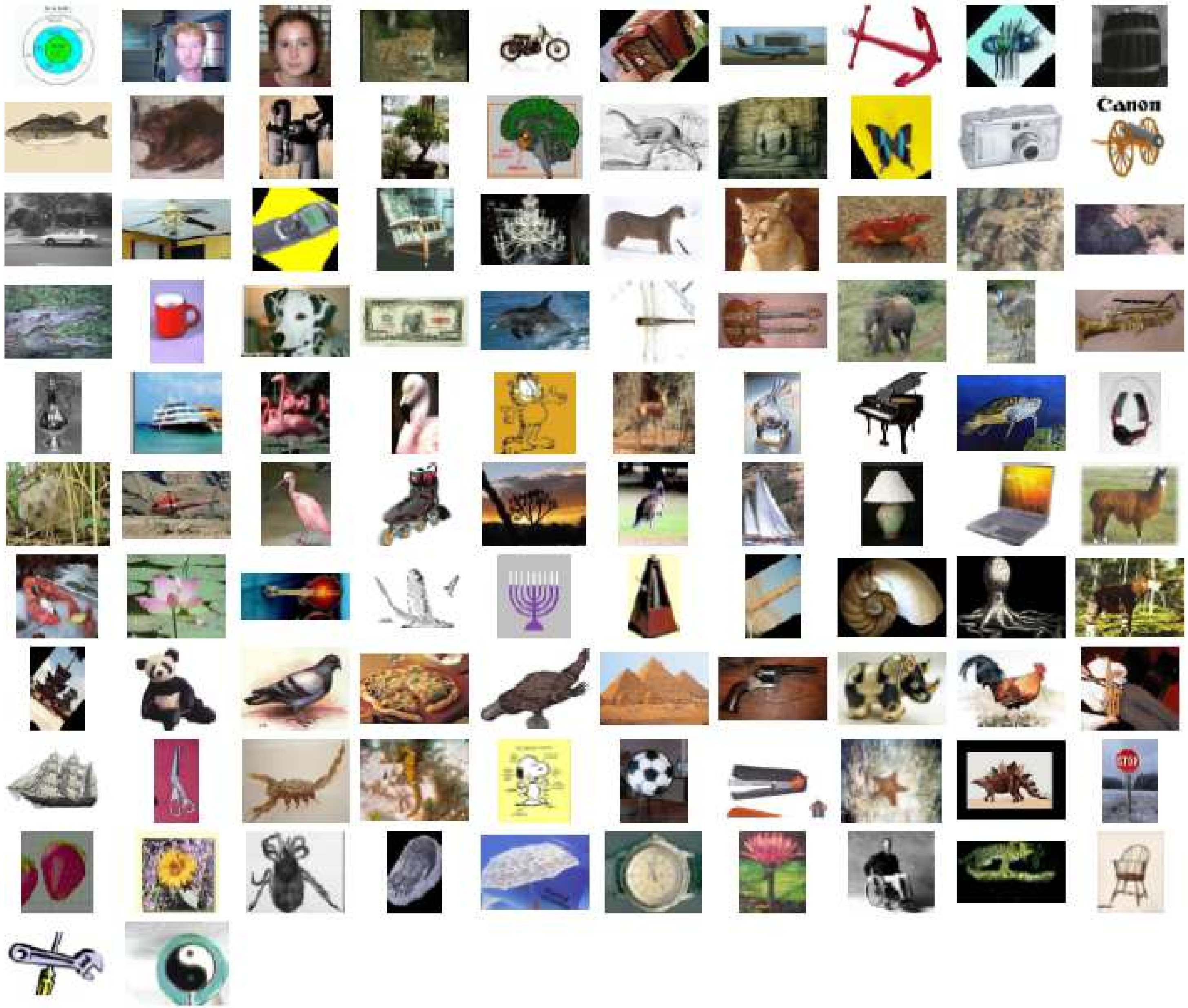
"Big data" revolution?
A new scientific context

- **Data everywhere**: size does not (always) matter
- **Science and industry**
- **Size and variety**
- **Learning from examples**
  - n observations in dimension d

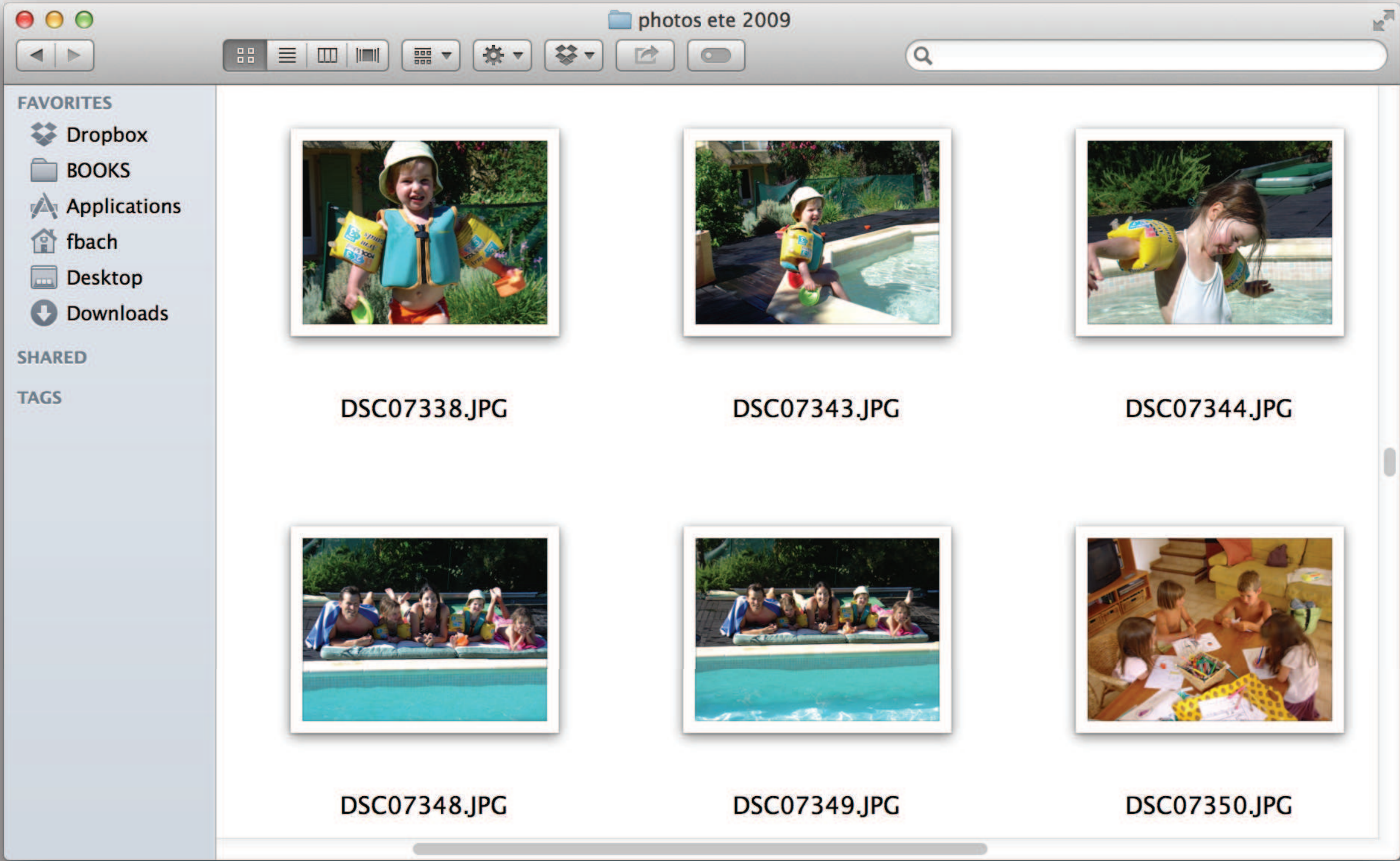
Search engines - Advertising



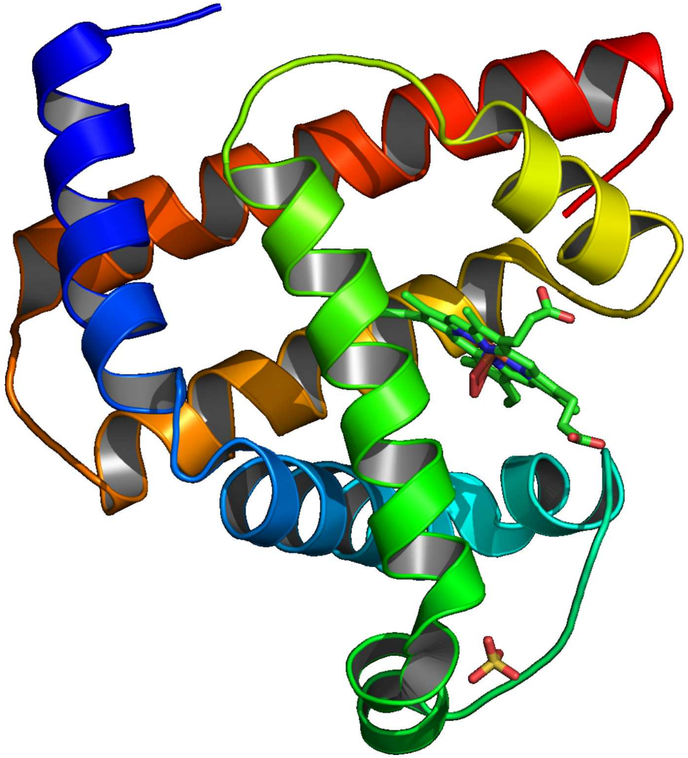
Visual object recognition



Personal photos



Bioinformatics



- **Protein**: Crucial elements of cell life

- **Massive data**: 2 millions for humans

- **Complex data**

Context
Machine learning for "big data"

- **Large-scale machine learning**: **large d, large n**
  - d : dimension of each observation (input)
  - n : number of observations

- **Examples**: computer vision, bioinformatics, advertising

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- **Ideal running-time complexity**: $O(dn)$

Context
Machine learning for "big data"

- **Large-scale machine learning**: **large d, large n**
  - d : dimension of each observation (input)
  - n : number of observations
- **Examples**: computer vision, bioinformatics, advertising
- **Ideal running-time complexity**: $O(dn)$
- **Going back to simple methods**
  - Stochastic gradient methods (Robbins and Monro, 1951)
  - Mixing statistics and optimization

Outline - I

1. **Introduction**

   - Large-scale machine learning and optimization
   - Classes of functions (convex, smooth, etc.)
   - Traditional statistical analysis through Rademacher complexity

2. **Classical methods for convex optimization**

   - Smooth optimization (gradient descent, Newton method)
   - Non-smooth optimization (subgradient descent)

3. **Classical stochastic approximation** (not covered)

   - Robbins-Monro algorithm (1951)

Outline - II

4. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds

5. **Smooth stochastic approximation algorithms**
   - Non-asymptotic analysis for smooth functions
   - Least-squares regression without decaying step-sizes

6. **Finite data sets**
   - Gradient methods with exponential convergence rates

Supervised machine learning

- **Data**: n observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$, $i = 1, \dots, n$, **i.i.d.**
- Prediction as a linear function $\theta^\top \Phi(x)$ of features $\Phi(x) \in \mathbb{R}^d$
- **(regularized) empirical risk minimization**: find $\hat{\theta}$ solution of

$$\min_{\theta \in \mathbb{R}^d} \quad \frac{1}{n} \sum_{i=1}^{n} \ell\big(y_i, \theta^\top \Phi(x_i)\big) \quad + \quad \mu \Omega(\theta)$$

convex data fitting term + regularizer

Usual losses

- **Regression**: $y \in \mathbb{R}$, prediction $\hat{y} = \theta^\top \Phi(x)$
  - quadratic loss $\frac{1}{2}(y - \hat{y})^2 = \frac{1}{2}(y - \theta^\top \Phi(x))^2$

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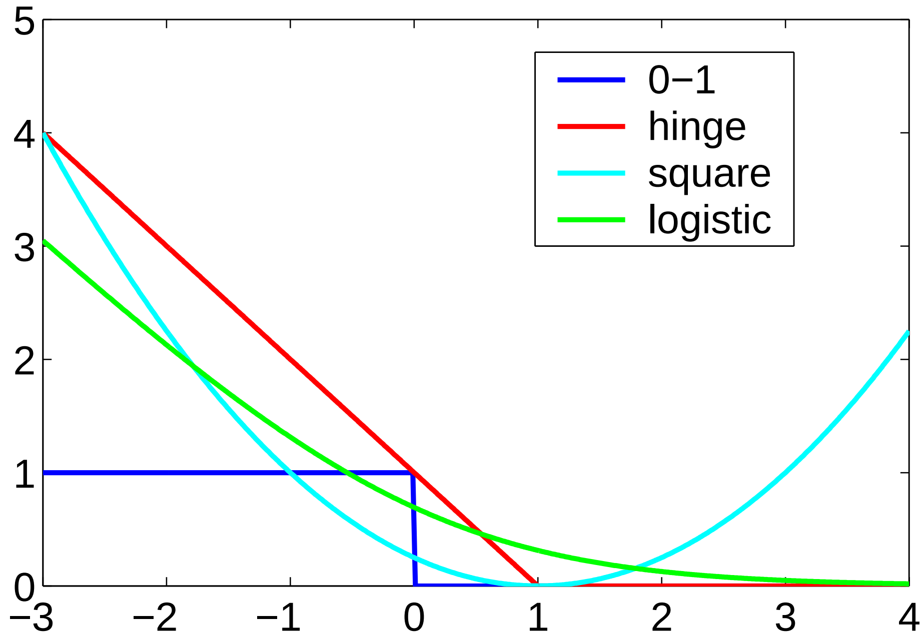
- **Classification** : $y \in \{-1, 1\}$, prediction $\hat{y} = \text{sign}(\theta^\top \Phi(x))$
  – loss of the form $\ell(y\,\theta^\top \Phi(x))$
  – "True" 0-1 loss: $\ell(y\,\theta^\top \Phi(x)) = 1_{y\,\theta^\top \Phi(x) < 0}$
  – Usual convex losses:



Main motivating examples

- **Support vector machine** (hinge loss): non-smooth

$$\ell(Y, \theta^\top \Phi(X)) = \max\{1 - Y\theta^\top \Phi(X), 0\}$$

- **Logistic regression**: smooth

$$\ell(Y, \theta^\top \Phi(X)) = \log(1 + \exp(-Y\theta^\top \Phi(X)))$$

- **Least-squares regression**

$$\ell(Y, \theta^\top \Phi(X)) = \frac{1}{2}(Y - \theta^\top \Phi(X))^2$$

- **Structured output regression**
  - See Tsochantaridis et al. (2005); Lacoste-Julien et al. (2013)

Usual regularizers

- **Main goal**: avoid overfitting

- **(squared) Euclidean norm**: $\|\theta\|_2^2 = \sum_{j=1}^{d} |\theta_j|^2$
  - Numerically well-behaved
  - Representer theorem and kernel methods : $\theta = \sum_{i=1}^{n} \alpha_i \Phi(x_i)$
  - See, e.g., Schölkopf and Smola (2001); Shawe-Taylor and Cristianini (2004)

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- **Sparsity-inducing norms**
  - Main example: ℓ_1-norm $\|\theta\|_1 = \sum_{j=1}^{d} |\theta_j|$
  - Perform model selection as well as regularization
  - Non-smooth optimization and structured sparsity
  - See, e.g., Bach, Jenatton, Mairal, and Obozinski (2012b,a)

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convex data fitting term $+$ regularizer

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convex data fitting term + regularizer

- Empirical risk: $\hat{f}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, \theta^\top \Phi(x_i))$ training cost
- Expected risk: $f(\theta) = \mathbb{E}_{(x,y)} \ell(y, \theta^\top \Phi(x))$ testing cost
- **Two fundamental questions**: (1) computing $\hat{\theta}$ and (2) analyzing $\hat{\theta}$

Supervised machine learning

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  - **May be tackled simultaneously**

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$$\min_{\theta \in \mathbb{R}^d} \quad \frac{1}{n} \sum_{i=1}^{n} \ell\big(y_i, \theta^\top \Phi(x_i)\big) \text{ such that } \Omega(\theta) \leqslant D$$

convex data fitting term + constraint

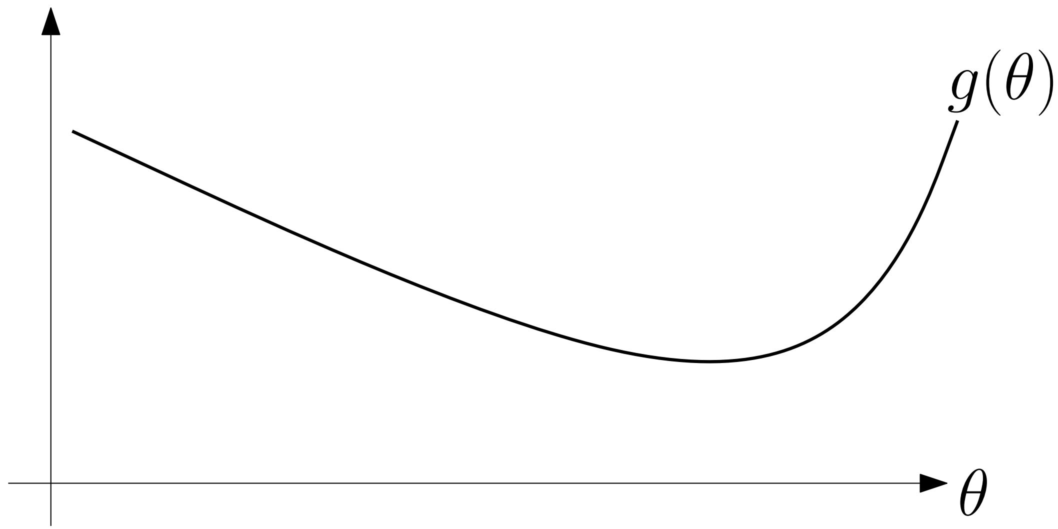
- Empirical risk: $\hat{f}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, \theta^\top \Phi(x_i))$ training cost
- Expected risk: $f(\theta) = \mathbb{E}_{(x,y)} \ell(y, \theta^\top \Phi(x))$ testing cost
- **Two fundamental questions**: (1) computing $\hat{\theta}$ and (2) analyzing $\hat{\theta}$
  - May be tackled simultaneously

General assumptions

- **Data**: n observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$, $i = 1, \dots, n$, **i.i.d.**
- Bounded features $\Phi(x) \in \mathbb{R}^d$: $\|\Phi(x)\|_2 \leqslant R$
- Empirical risk: $\hat{f}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, \theta^\top \Phi(x_i))$ training cost
- Expected risk: $f(\theta) = \mathbb{E}_{(x,y)} \ell(y, \theta^\top \Phi(x))$ testing cost
- Loss for a single observation: $f_i(\theta) = \ell(y_i, \theta^\top \Phi(x_i))$
  $\Rightarrow \forall i, \ f(\theta) = \mathbb{E} f_i(\theta)$
- **Properties of** $f_i, f, \hat{f}$
  - Convex on $\mathbb{R}^d$
  - Additional regularity assumptions: Lipschitz-continuity, smoothness and strong convexity

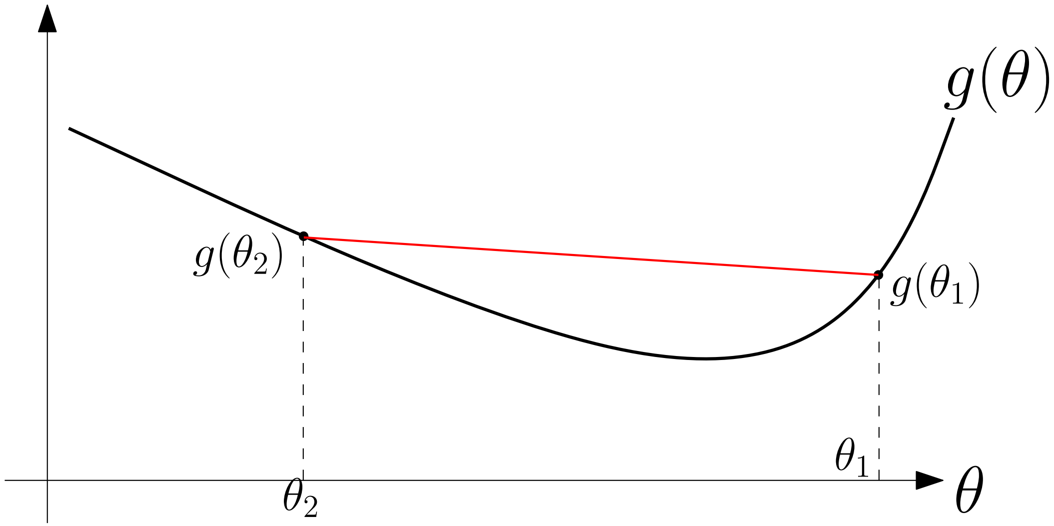
Convexity

- **Global definitions**



Convexity

- **Global definitions (full domain)**



  – Not assuming differentiability:

$$\forall \theta_1, \theta_2, \alpha \in [0,1], \quad g(\alpha\theta_1 + (1-\alpha)\theta_2) \leqslant \alpha g(\theta_1) + (1-\alpha) g(\theta_2)$$

Convexity

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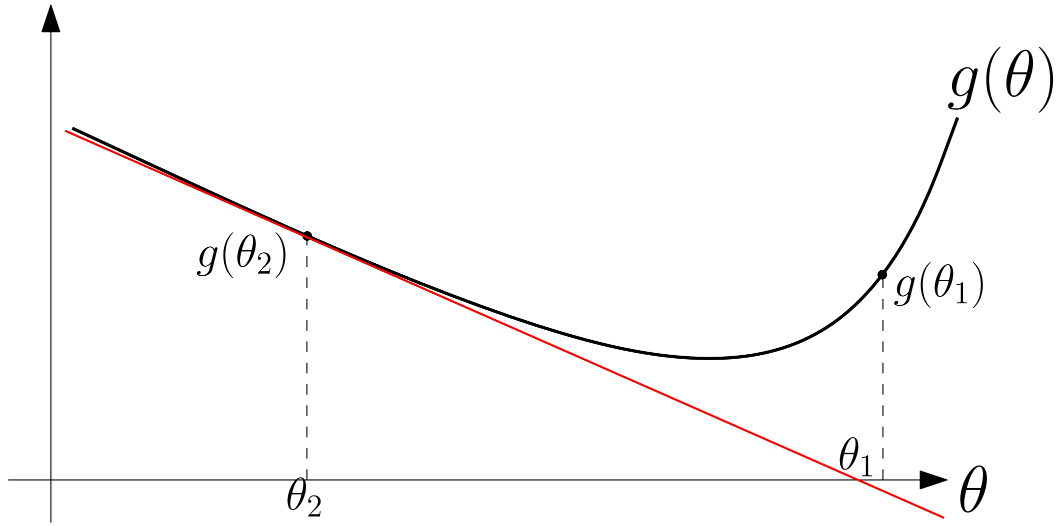


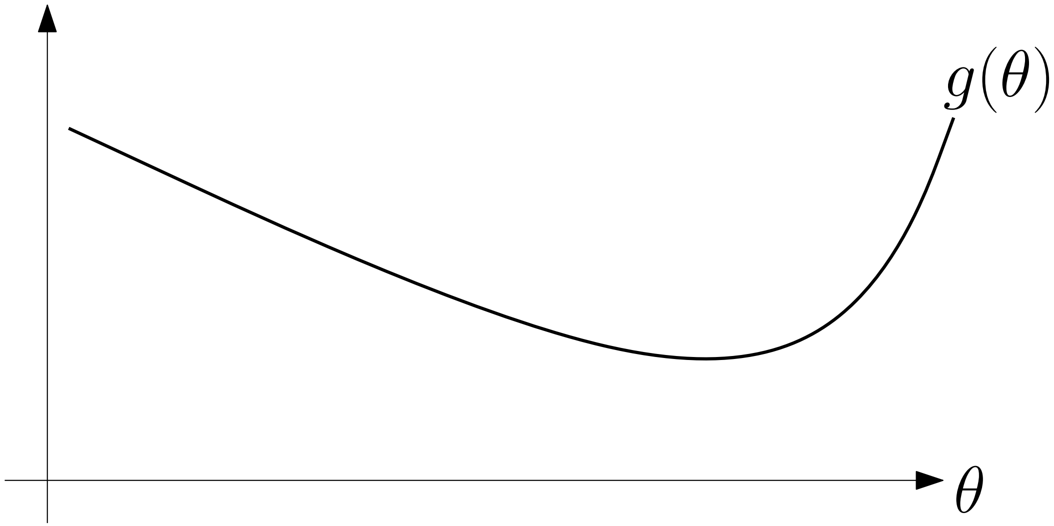
  – Assuming differentiability:

$$\forall \theta_1, \theta_2, \quad g(\theta_1) \geqslant g(\theta_2) + g'(\theta_2)^\top (\theta_1 - \theta_2)$$

- **Extensions to all functions with subgradients / subdifferential**

Convexity

- **Global definitions (full domain)**



- **Local definitions**
  - Twice differentiable functions
  - $\forall \theta,\ g''(\theta) \succcurlyeq 0$ (positive semi-definite Hessians)

Convexity

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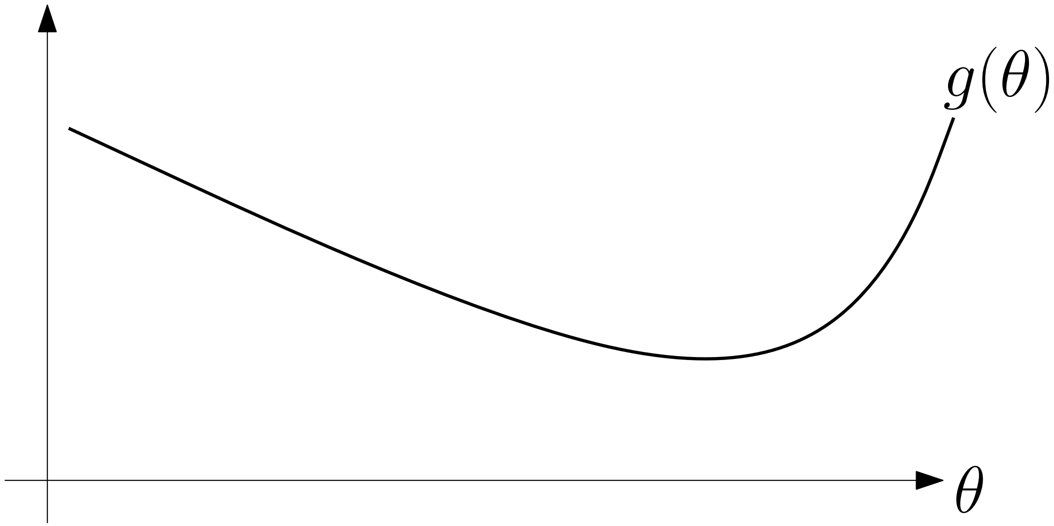

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- **Why convexity?**

Why convexity?

- **Local minimum = global minimum**
  - Optimality condition (non-smooth): $0 \in \partial g(\theta)$
  - Optimality condition (smooth): $g'(\theta) = 0$

- **Convex duality**
  - See Boyd and Vandenberghe (2003)

- **Recognizing convex problems**
  - See Boyd and Vandenberghe (2003)

Lipschitz continuity

- **Bounded gradients of g ($\Leftrightarrow$ Lipschitz-continuity)**: the function g if convex, differentiable and has (sub)gradients uniformly bounded by B on the ball of center 0 and radius D:

$$\forall \theta \in \mathbb{R}^d, \|\theta\|_2 \leqslant D \Rightarrow \|g'(\theta)\|_2 \leqslant B$$

$$\Leftrightarrow$$

$$\forall \theta, \theta' \in \mathbb{R}^d, \|\theta\|_2, \|\theta'\|_2 \leqslant D \Rightarrow |g(\theta) - g(\theta')| \leqslant B\|\theta - \theta'\|_2$$

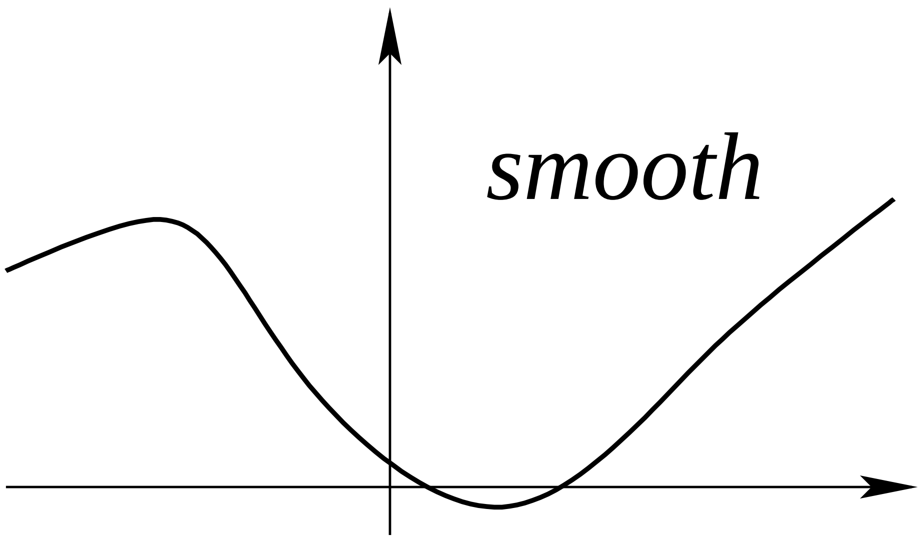
- **Machine learning**
  - with $g(\theta) = \frac{1}{n}\sum_{i=1}^n \ell(y_i, \theta^\top \Phi(x_i))$
  - G-Lipschitz loss and R-bounded data: $B = GR$

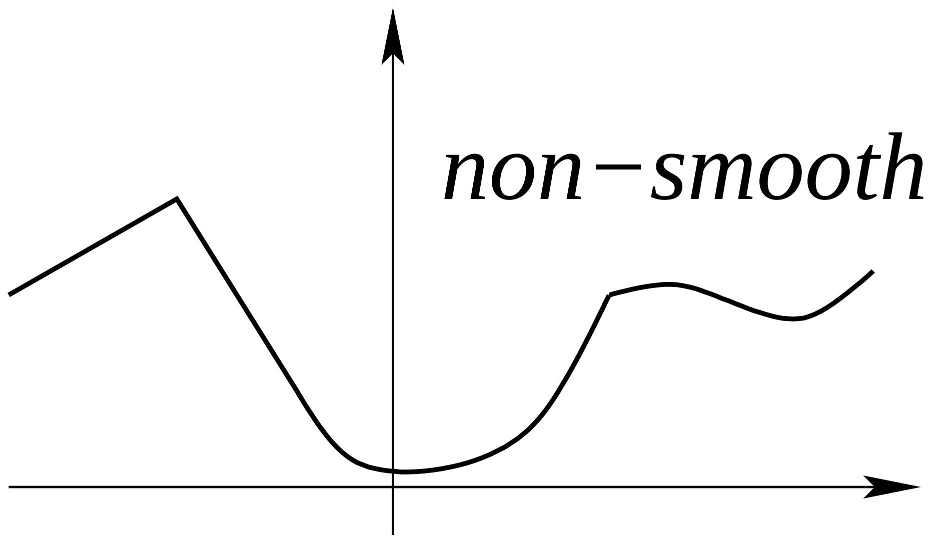
Smoothness and strong convexity

- A function $g : \mathbb{R}^d \to \mathbb{R}$ is L-smooth if and only if it is differentiable and its gradient is L-Lipschitz-continuous

$$\forall \theta_1, \theta_2 \in \mathbb{R}^d,\ \|g'(\theta_1) - g'(\theta_2)\|_2 \leqslant L\|\theta_1 - \theta_2\|_2$$

- If g is twice differentiable: $\forall \theta \in \mathbb{R}^d,\ g''(\theta) \preccurlyeq L \cdot Id$





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  - Hessian $\approx$ covariance matrix $\frac{1}{n}\sum_{i=1}^n \Phi(x_i)\Phi(x_i)^\top$
  - $L_{\rm loss}$-smooth loss and R-bounded data: $L = L_{\rm loss} R^2$

Smoothness and strong convexity

- A function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is μ-strongly convex if and only if

$$\forall \theta_1, \theta_2 \in \mathbb{R}^d, \;\; g(\theta_1) \geqslant g(\theta_2) + g'(\theta_2)^\top(\theta_1 - \theta_2) + \frac{\mu}{2}\|\theta_1 - \theta_2\|_2^2$$

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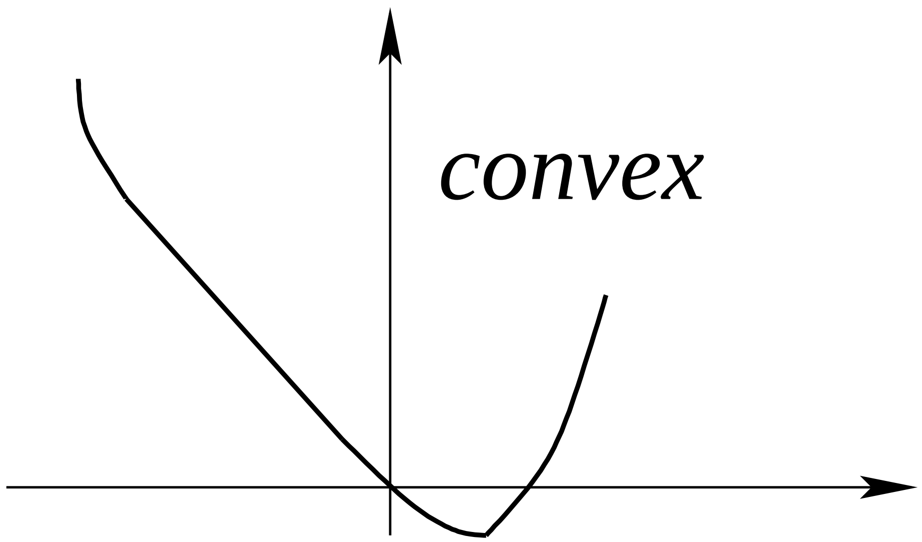

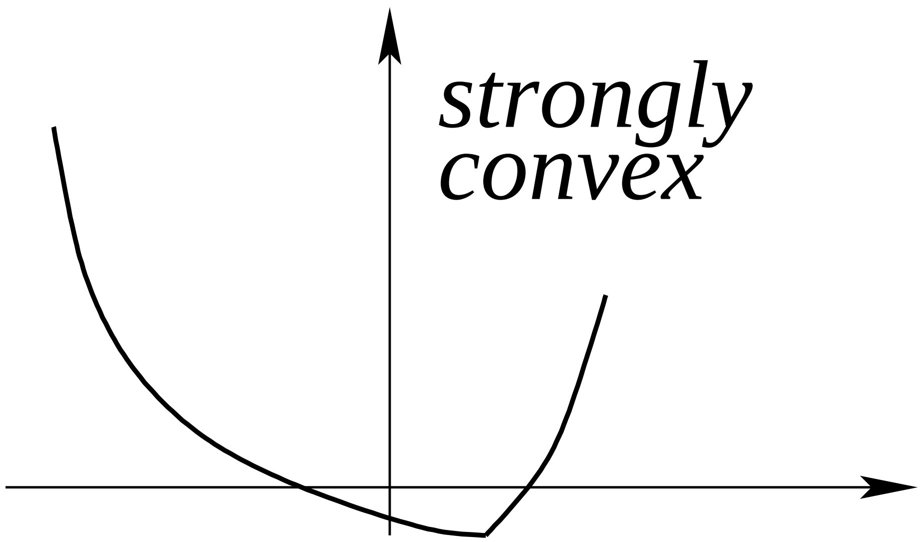

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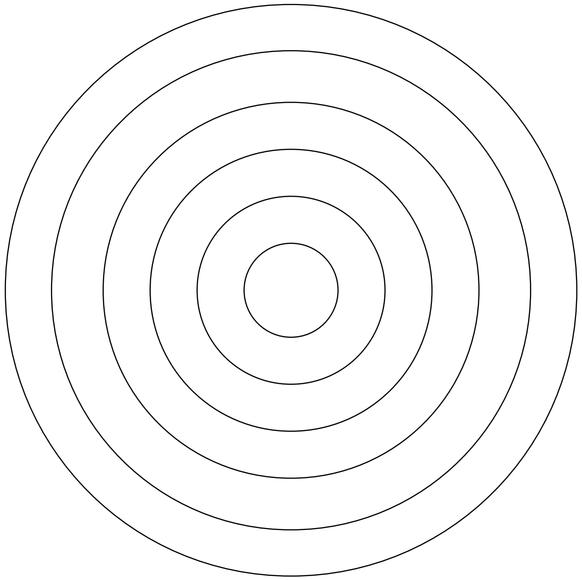
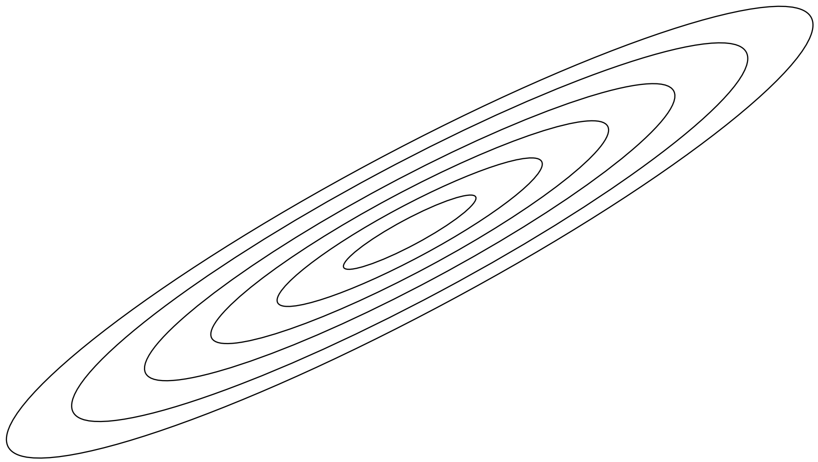


(large μ/L)



(small μ/L)

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- If g is twice differentiable: $\forall \theta \in \mathbb{R}^d, \; g''(\theta) \succcurlyeq \mu \cdot \mathrm{Id}$

- **Machine learning**
  - with $g(\theta) = \frac{1}{n} \sum_{i=1}^n \ell(y_i, \theta^\top \Phi(x_i))$
  - Hessian $\approx$ covariance matrix $\frac{1}{n} \sum_{i=1}^n \Phi(x_i) \Phi(x_i)^\top$
  - Data with invertible covariance matrix (low correlation/dimension)

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  - Data with invertible covariance matrix (low correlation/dimension)

- **Adding regularization by** $\frac{\mu}{2} \|\theta\|^2$
  - creates additional bias unless μ is small

Summary of smoothness/convexity assumptions

- **Bounded gradients of g (Lipschitz-continuity)**: the function g if convex, differentiable and has (sub)gradients uniformly bounded by B on the ball of center 0 and radius D:

$$\forall \theta \in \mathbb{R}^d, \|\theta\|_2 \leqslant D \Rightarrow \|g'(\theta)\|_2 \leqslant B$$

- **Smoothness of g**: the function g is convex, differentiable with L-Lipschitz-continuous gradient g' (e.g., bounded Hessians):

$$\forall \theta_1, \theta_2 \in \mathbb{R}^d, \quad \|g'(\theta_1) - g'(\theta_2)\|_2 \leqslant L\|\theta_1 - \theta_2\|_2$$

- **Strong convexity of g**: The function g is strongly convex with respect to the norm $\|\cdot\|$, with convexity constant $\mu > 0$:

$$\forall \theta_1, \theta_2 \in \mathbb{R}^d, \; g(\theta_1) \geqslant g(\theta_2) + g'(\theta_2)^\top(\theta_1 - \theta_2) + \frac{\mu}{2}\|\theta_1 - \theta_2\|_2^2$$

Analysis of empirical risk minimization

- **Approximation and estimation errors**: $\Theta = \{\theta \in \mathbb{R}^d, \Omega(\theta) \leqslant D\}$

$$f(\hat{\theta}) - \min_{\theta \in \mathbb{R}^d} f(\theta) \ = \ \left[f(\hat{\theta}) - \min_{\theta \in \Theta} f(\theta) \right] + \left[\min_{\theta \in \Theta} f(\theta) - \min_{\theta \in \mathbb{R}^d} f(\theta) \right]$$

Estimation error — Approximation error

  – NB: may replace $\min_{\theta \in \mathbb{R}^d} f(\theta)$ by best (non-linear) predictions

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Estimation error   Approximation error

1. **Uniform deviation bounds**, with $\boxed{\hat{\theta} \in \arg\min_{\theta \in \Theta} \hat{f}(\theta)}$

$$f(\hat{\theta}) - \min_{\theta \in \Theta} f(\theta) \leqslant 2 \cdot \sup_{\theta \in \Theta} |f(\theta) - \hat{f}(\theta)|$$

   – Typically slow rate $O\big(1/\sqrt{n}\big)$

2. **More refined concentration results** with faster rates $O(1/n)$

Slow rate for supervised learning

- **Assumptions** (f is the expected risk, $\hat{f}$ the empirical risk)
  - $\Omega(\theta) = \|\theta\|_2$ (Euclidean norm)
  - "Linear" predictors: $\theta(x) = \theta^\top \Phi(x)$, with $\|\Phi(x)\|_2 \leqslant R$ a.s.
  - G-Lipschitz loss: f and $\hat{f}$ are GR-Lipschitz on $\Theta = \{\|\theta\|_2 \leqslant D\}$
  - No assumptions regarding convexity

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  - No assumptions regarding convexity

- With probability greater than $1 - \delta$

$$\sup_{\theta \in \Theta} |\hat{f}(\theta) - f(\theta)| \leqslant \frac{\ell_0 + GRD}{\sqrt{n}} \left[2 + \sqrt{2 \log \frac{2}{\delta}} \right]$$

- Expectated estimation error: $\mathbb{E}\big[\sup_{\theta \in \Theta} |\hat{f}(\theta) - f(\theta)|\big] \leqslant \dfrac{4\ell_0 + 4GRD}{\sqrt{n}}$

- Using Rademacher averages (see, e.g., Boucheron et al., 2005)

- **Lipschitz functions $\Rightarrow$ slow rate**

Motivation from mean estimation

- Estimator $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} z_i = \arg\min_{\theta \in \mathbb{R}} \frac{1}{2n}\sum_{i=1}^{n}(\theta - z_i)^2 = \hat{f}(\theta)$

- From before:
  - $f(\theta) = \frac{1}{2}\mathbb{E}(\theta - z)^2 = \frac{1}{2}(\theta - \mathbb{E}z)^2 + \frac{1}{2}\operatorname{var}(z) = \hat{f}(\theta) + O(1/\sqrt{n})$
  - $f(\hat{\theta}) = \frac{1}{2}(\hat{\theta} - \mathbb{E}z)^2 + \frac{1}{2}\operatorname{var}(z) = f(\mathbb{E}z) + O(1/\sqrt{n})$

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- Estimator $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} z_i = \arg\min_{\theta\in\mathbb{R}} \frac{1}{2n}\sum_{i=1}^{n}(\theta - z_i)^2 = \hat{f}(\theta)$
- From before:
  - $f(\theta) = \frac{1}{2}\mathbb{E}(\theta - z)^2 = \frac{1}{2}(\theta - \mathbb{E}z)^2 + \frac{1}{2}\operatorname{var}(z) = \hat{f}(\theta) + O(1/\sqrt{n})$
  - $f(\hat{\theta}) = \frac{1}{2}(\hat{\theta} - \mathbb{E}z)^2 + \frac{1}{2}\operatorname{var}(z) = f(\mathbb{E}z) + O(1/\sqrt{n})$
- More refined/direct bound:

$$\begin{aligned} f(\hat{\theta}) - f(\mathbb{E}z) &= \frac{1}{2}(\hat{\theta} - \mathbb{E}z)^2 \\ \mathbb{E}\big[f(\hat{\theta}) - f(\mathbb{E}z)\big] &= \frac{1}{2}\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} z_i - \mathbb{E}z\right)^2 = \frac{1}{2n}\operatorname{var}(z) \end{aligned}$$

- Bound only at $\hat{\theta}$ + strong convexity (instead of uniform bound)

Fast rate for supervised learning

- **Assumptions** (f is the expected risk, $\hat{f}$ the empirical risk)
  - Same as before (bounded features, Lipschitz loss)
  - Regularized risks: $f^\mu(\theta) = f(\theta) + \frac{\mu}{2}\|\theta\|_2^2$ and $\hat{f}^\mu(\theta) = \hat{f}(\theta) + \frac{\mu}{2}\|\theta\|_2^2$
  - Convexity
- For any $a > 0$, with probability greater than $1 - \delta$, for all $\theta \in \mathbb{R}^d$,

$$f^\mu(\hat{\theta}) - \min_{\eta \in \mathbb{R}^d} f^\mu(\eta) \leqslant \frac{8(1 + \frac{1}{a})G^2R^2(32 + \log\frac{1}{\delta})}{\mu n}$$

- Results from Sridharan, Srebro, and Shalev-Shwartz (2008)
  - see also Boucheron and Massart (2011) and references therein
- **Strongly convex functions $\Rightarrow$ fast rate**
  - Warning: μ should decrease with n to reduce approximation error

Outline - I

1. **Introduction**

   - Large-scale machine learning and optimization
   - Classes of functions (convex, smooth, etc.)
   - Traditional statistical analysis through Rademacher complexity

2. **Classical methods for convex optimization**

   - Smooth optimization (gradient descent, Newton method)
   - Non-smooth optimization (subgradient descent)

3. **Classical stochastic approximation** (not covered)

   - Robbins-Monro algorithm (1951)

Outline - II

4. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds

5. **Smooth stochastic approximation algorithms**
   - Non-asymptotic analysis for smooth functions
   - Least-squares regression without decaying step-sizes

6. **Finite data sets**
   - Gradient methods with exponential convergence rates

Complexity results in convex optimization

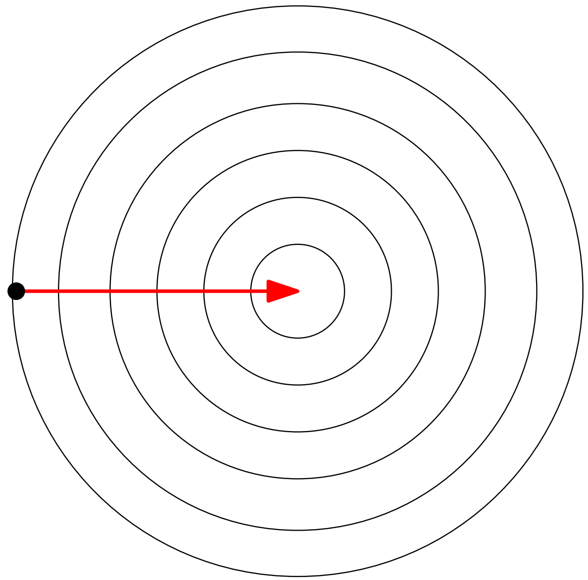
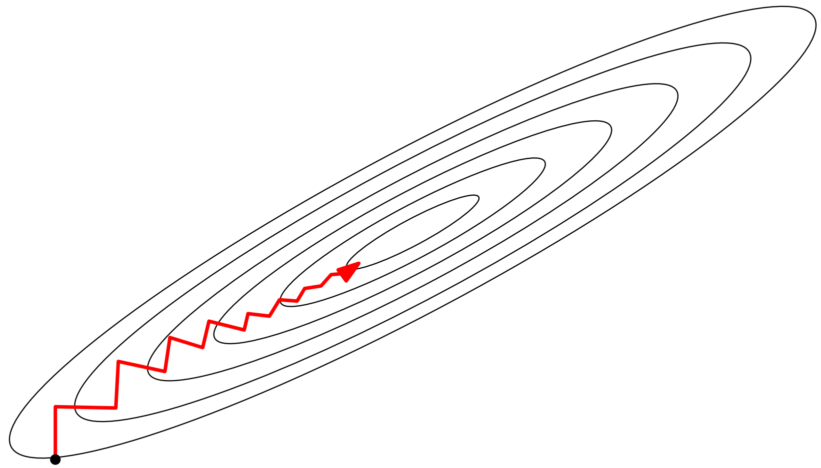
- **Assumption**: g convex on $\mathbb{R}^d$
- **Classical generic algorithms**
  - Gradient descent and accelerated gradient descent
  - Newton method
  - Subgradient method and ellipsoid algorithm
- **Key additional properties of g**
  - Lipschitz continuity, smoothness or strong convexity
- **Key insight from Bottou and Bousquet (2008)**
  - In machine learning, no need to optimize below estimation error
- **Key references**: Nesterov (2004), Bubeck (2015)

(smooth) gradient descent

- **Assumptions**
  - g convex with L-Lipschitz-continuous gradient (e.g., L-smooth)

- **Algorithm**:

$$\theta_t = \theta_{t-1} - \frac{1}{L} g'(\theta_{t-1})$$

(smooth) gradient descent - strong convexity

- **Assumptions**
  - g convex with L-Lipschitz-continuous gradient (e.g., L-smooth)
  - g μ-strongly convex
- **Algorithm**:
$$\theta_t = \theta_{t-1} - \frac{1}{L} g'(\theta_{t-1})$$
- **Bound**:
$$g(\theta_t) - g(\theta_*) \leqslant (1 - \mu/L)^t \big[g(\theta_0) - g(\theta_*) \big]$$
- Three-line proof
- **Line search, steepest descent or constant step-size**

(smooth) gradient descent - slow rate

- **Assumptions**
  - g convex with L-Lipschitz-continuous gradient (e.g., L-smooth)
  - Minimum attained at θ_*

- **Algorithm**:

$$\theta_t = \theta_{t-1} - \frac{1}{L} g'(\theta_{t-1})$$

- **Bound**:

$$g(\theta_t) - g(\theta_*) \leqslant \frac{2L\|\theta_0 - \theta_*\|^2}{t+4}$$

- Four-line proof

- **Adaptivity of gradient descent to problem difficulty**

- Not best possible convergence rates after $O(d)$ iterations

Gradient descent - Proof for quadratic functions

- Quadratic convex function: $g(\theta) = \frac{1}{2}\theta^\top H\theta - c^\top\theta$
  - μ and L are smallest largest eigenvalues of H
  - Global optimum $\theta_* = H^{-1}c$ (or $H^\dagger c$)

- Gradient descent:

$$\begin{aligned}
\theta_t &= \theta_{t-1} - \frac{1}{L}(H\theta - c) = \theta_{t-1} - \frac{1}{L}(H\theta - H\theta_*) \\
\theta_t - \theta_* &= (I - \frac{1}{L}H)(\theta_{t-1} - \theta_*) = (I - \frac{1}{L}H)^t(\theta_0 - \theta_*)
\end{aligned}$$

- **Strong convexity** $\mu > 0$: eigenvalues of $(I - \frac{1}{L}H)^t$ in $[0, (1 - \frac{\mu}{L})^t]$
  - Convergence of iterates: $\|\theta_t - \theta_*\|^2 \leqslant (1 - \mu/L)^{2t}\|\theta_0 - \theta_*\|^2$
  - Function values: $g(\theta_t) - g(\theta_*) \leqslant (1 - \mu/L)^{2t}\big[g(\theta_0) - g(\theta_*)\big]$

Gradient descent - Proof for quadratic functions

- Quadratic convex function: $g(\theta) = \frac{1}{2}\theta^\top H\theta - c^\top\theta$
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- Gradient descent:

$$\begin{aligned}\theta_t &= \theta_{t-1} - \frac{1}{L}(H\theta - c) = \theta_{t-1} - \frac{1}{L}(H\theta - H\theta_*)\\ \theta_t - \theta_* &= (I - \frac{1}{L}H)(\theta_{t-1} - \theta_*) = (I - \frac{1}{L}H)^t(\theta_0 - \theta_*)\end{aligned}$$

- **Convexity** $\mu = 0$: eigenvalues of $(I - \frac{1}{L}H)^t$ in $[0, 1]$
  - No convergence of iterates: $\|\theta_t - \theta_*\|^2 \leqslant \|\theta_0 - \theta_*\|^2$
  - Function values: $g(\theta_t) - g(\theta_*) \leqslant \max_{v\in[0,L]} v(1 - v/L)^{2t}\|\theta_0 - \theta_*\|^2$
    $$g(\theta_t) - g(\theta_*) \leqslant \frac{L}{t}\|\theta_0 - \theta_*\|^2$$

Accelerated gradient methods (Nesterov, 1983)

- **Assumptions**
  - g convex with L-Lipschitz-cont. gradient , min. attained at θ_*
- **Algorithm**:

$$\begin{aligned}\theta_t &= \eta_{t-1} - \frac{1}{L} g'(\eta_{t-1}) \\ \eta_t &= \theta_t + \frac{t-1}{t+2}(\theta_t - \theta_{t-1})\end{aligned}$$

- **Bound**:

$$g(\theta_t) - g(\theta_*) \leqslant \frac{2L\|\theta_0 - \theta_*\|^2}{(t+1)^2}$$

- Ten-line proof (see, e.g., Schmidt, Le Roux, and Bach, 2011)
- Not improvable
- Extension to strongly-convex functions

Accelerated gradient methods - strong convexity

- **Assumptions**
  - g convex with L-Lipschitz-cont. gradient , min. attained at θ_*
  - g μ-strongly convex
- **Algorithm**:

$$\begin{aligned} \theta_t &= \eta_{t-1} - \frac{1}{L} g'(\eta_{t-1}) \\ \eta_t &= \theta_t + \frac{1-\sqrt{\mu/L}}{1+\sqrt{\mu/L}}(\theta_t - \theta_{t-1}) \end{aligned}$$

- **Bound**: $g(\theta_t) - f(\theta_*) \leqslant L\|\theta_0 - \theta_*\|^2 (1 - \sqrt{\mu/L})^t$
  - Ten-line proof (see, e.g., Schmidt, Le Roux, and Bach, 2011)
  - Not improvable
  - Relationship with conjugate gradient for quadratic functions

Optimization for sparsity-inducing norms
(see Bach, Jenatton, Mairal, and Obozinski, 2012b)

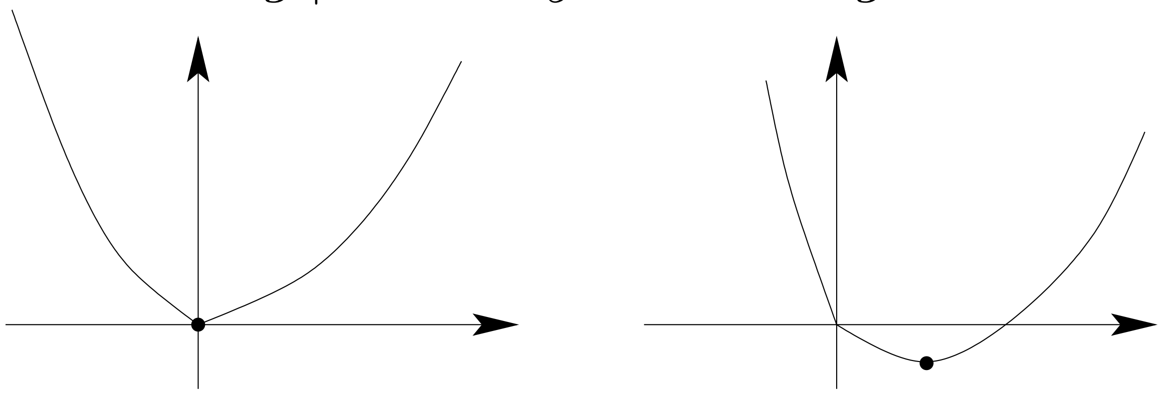
- Gradient descent as a **proximal method** (differentiable functions)
  - $\theta_{t+1} = \arg\min_{\theta \in \mathbb{R}^d} f(\theta_t) + (\theta - \theta_t)^\top \nabla f(\theta_t) + \frac{L}{2} \|\theta - \theta_t\|_2^2$
  - $\theta_{t+1} = \theta_t - \frac{1}{L} \nabla f(\theta_t)$

Optimization for sparsity-inducing norms (see Bach, Jenatton, Mairal, and Obozinski, 2012b)

- Gradient descent as a **proximal method** (differentiable functions)
  - $\theta_{t+1} = \arg\min_{\theta \in \mathbb{R}^d} f(\theta_t) + (\theta - \theta_t)^\top \nabla f(\theta_t) + \frac{L}{2}\|\theta - \theta_t\|_2^2$
  - $\theta_{t+1} = \theta_t - \frac{1}{L}\nabla f(\theta_t)$
- Problems of the form: $\boxed{\min_{\theta \in \mathbb{R}^d} f(\theta) + \mu\Omega(\theta)}$
  - $\theta_{t+1} = \arg\min_{\theta \in \mathbb{R}^d} f(\theta_t) + (\theta - \theta_t)^\top \nabla f(\theta_t) + \mu\Omega(\theta) + \frac{L}{2}\|\theta - \theta_t\|_2^2$
  - $\Omega(\theta) = \|\theta\|_1 \Rightarrow$ **Thresholded gradient descent**
- Similar convergence rates than smooth optimization
  - Acceleration methods (Nesterov, 2007; Beck and Teboulle, 2009)

Soft-thresholding for the ℓ_1-norm

- **Example 1**: quadratic problem in 1D, i.e. $\boxed{\min_{x\in\mathbb{R}} \frac{1}{2}x^2 - xy + \lambda|x|}$
- Piecewise quadratic function with a kink at zero
  - Derivative at $0+$: $g_+ = \lambda - y$ and $0-$: $g_- = -\lambda - y$



  - $x = 0$ is the solution iff $g_+ \geqslant 0$ and $g_- \leqslant 0$ (i.e., $|y| \leqslant \lambda$)
  - $x \geqslant 0$ is the solution iff $g_+ \leqslant 0$ (i.e., $y \geqslant \lambda$) $\Rightarrow x^* = y - \lambda$
  - $x \leqslant 0$ is the solution iff $g_- \leqslant 0$ (i.e., $y \leqslant -\lambda$) $\Rightarrow x^* = y + \lambda$

- Solution $\boxed{x^* = \text{sign}(y)(|y| - \lambda)_+}$ = soft thresholding

Soft-thresholding for the ℓ_1-norm

- **Example 1**: quadratic problem in 1D, i.e. $\boxed{\min_{x \in \mathbb{R}} \frac{1}{2}x^2 - xy + \lambda|x|}$
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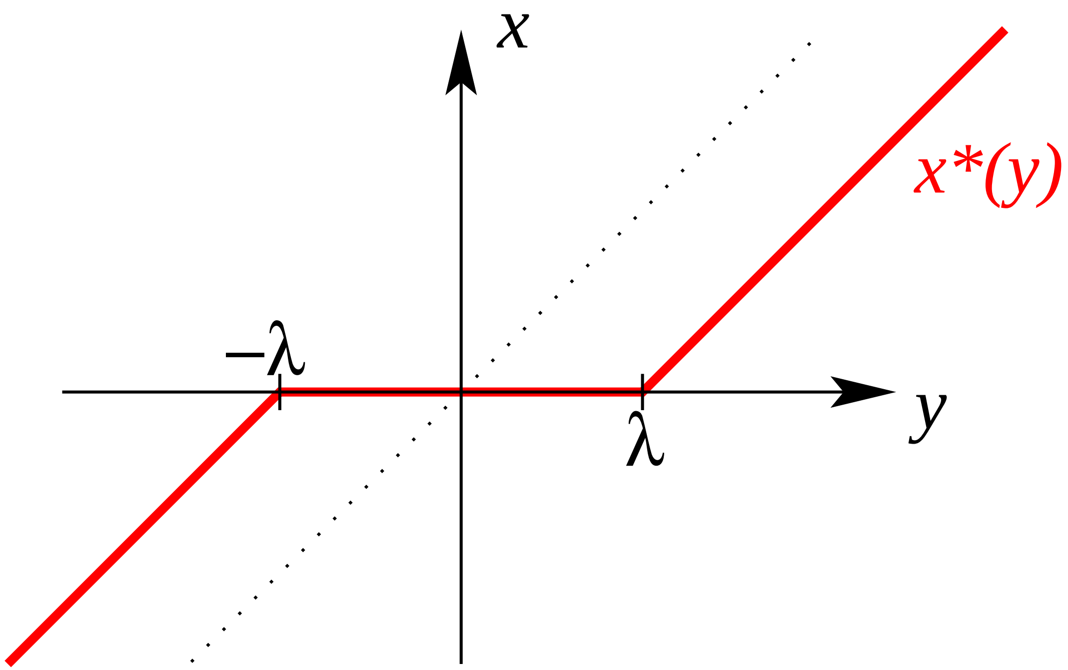

Newton method

- Given θ_{t-1}, minimize second-order Taylor expansion

$$\tilde{g}(\theta) = g(\theta_{t-1}) + g'(\theta_{t-1})^\top(\theta - \theta_{t-1}) + \frac{1}{2}(\theta - \theta_{t-1})^\top g''(\theta_{t-1})^\top(\theta - \theta_{t-1})$$

- **Expensive Iteration**: $\theta_t = \theta_{t-1} - g''(\theta_{t-1})^{-1} g'(\theta_{t-1})$
  - Running-time complexity: $O(d^3)$ in general

- **Quadratic convergence**: If $\|\theta_{t-1} - \theta_*\|$ small enough, for some constant C, we have

$$(C\|\theta_t - \theta_*\|) = (C\|\theta_{t-1} - \theta_*\|)^2$$

  - See Boyd and Vandenberghe (2003)

Summary: minimizing smooth convex functions

- **Assumption**: g convex
- **Gradient descent**: $\theta_t = \theta_{t-1} - \gamma_t \, g'(\theta_{t-1})$
  - $O(1/t)$ convergence rate for smooth convex functions
  - $O(e^{-t\mu/L})$ convergence rate for strongly smooth convex functions
  - Optimal rates $O(1/t^2)$ and $O(e^{-t\sqrt{\mu/L}})$
- **Newton method**: $\theta_t = \theta_{t-1} - f''(\theta_{t-1})^{-1} f'(\theta_{t-1})$
  - $O\big(e^{-\rho 2^t}\big)$ convergence rate

Summary: minimizing smooth convex functions

- **Assumption**: g convex
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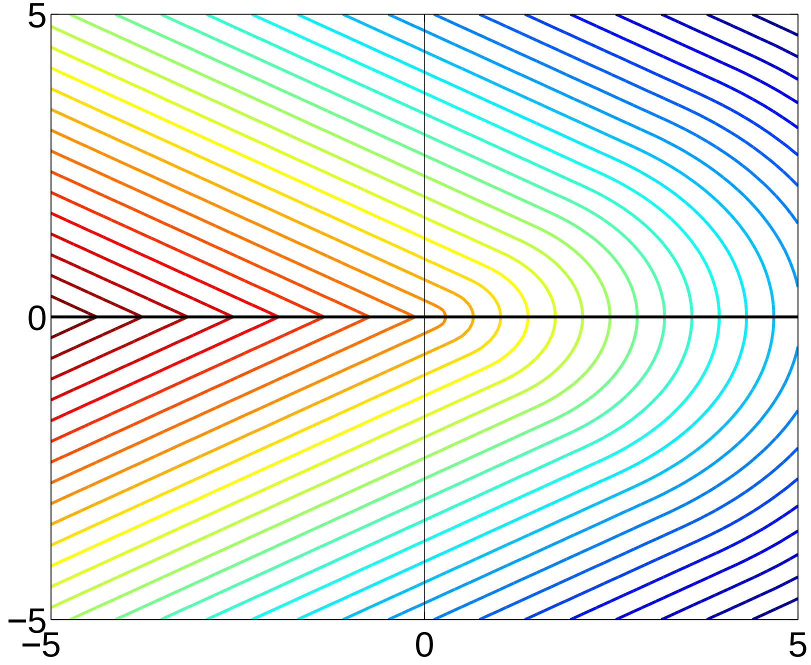
- **Newton method**: $\theta_t = \theta_{t-1} - f''(\theta_{t-1})^{-1} f'(\theta_{t-1})$
  - $O\big(e^{-\rho 2^t}\big)$ convergence rate
- **From smooth to non-smooth**
  - Subgradient method and ellipsoid (not covered)

Counter-example (Bertsekas, 1999)
Steepest descent for nonsmooth objectives

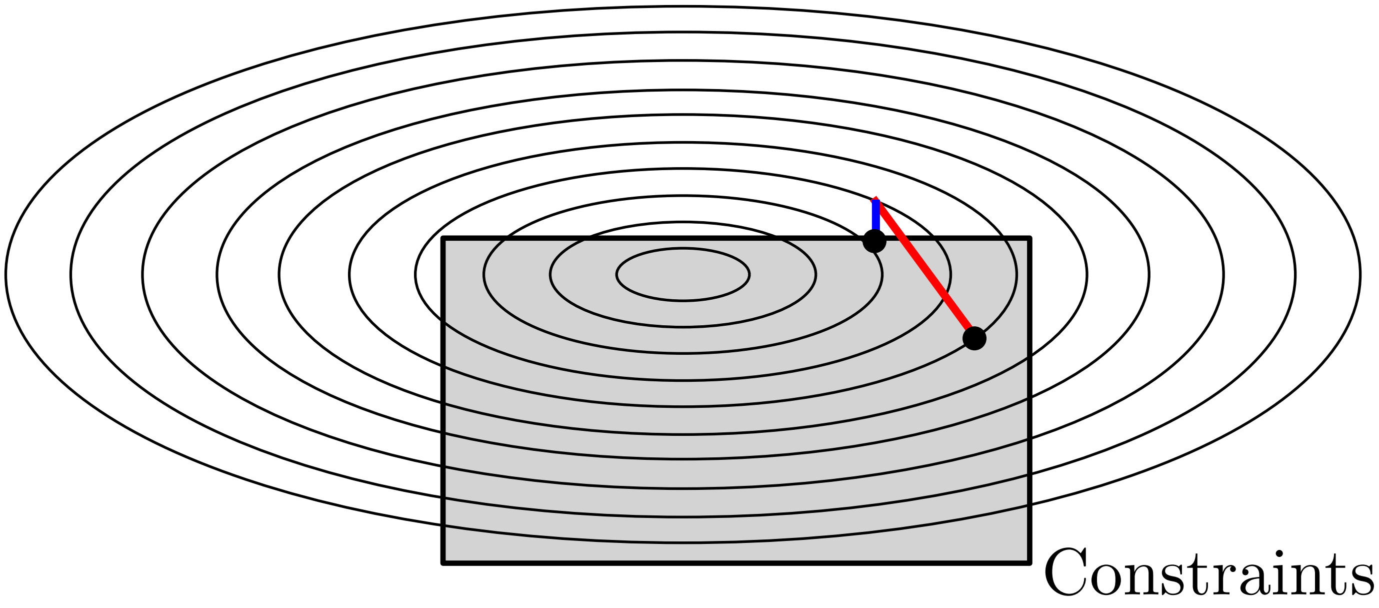
- $g(\theta_1, \theta_2) = \begin{cases} -5(9\theta_1^2 + 16\theta_2^2)^{1/2} & \text{if } \theta_1 > |\theta_2| \\ -(9\theta_1 + 16|\theta_2|)^{1/2} & \text{if } \theta_1 \leqslant |\theta_2| \end{cases}$

- Steepest descent starting from any θ such that $\theta_1 > |\theta_2| > (9/16)^2|\theta_1|$



Subgradient method/"descent" (Shor et al., 1985)

- **Assumptions**
  - g convex and B-Lipschitz-continuous on $\{\|\theta\|_2 \leqslant D\}$
- **Algorithm**: $\theta_t = \Pi_D\Big(\theta_{t-1} - \frac{2D}{B\sqrt{t}} g'(\theta_{t-1})\Big)$
  - Π_D : orthogonal projection onto $\{\|\theta\|_2 \leqslant D\}$



Subgradient method/“descent” (Shor et al., 1985)

- **Assumptions**
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- **Algorithm**: $\theta_t = \Pi_D\Big(\theta_{t-1} - \frac{2D}{B\sqrt{t}} g'(\theta_{t-1})\Big)$
  - Π_D : orthogonal projection onto $\{\|\theta\|_2 \leqslant D\}$
- **Bound**:

$$g\bigg(\frac{1}{t}\sum_{k=0}^{t-1}\theta_k\bigg) - g(\theta_*) \leqslant \frac{2DB}{\sqrt{t}}$$

- Three-line proof
- Best possible convergence rate after $O(d)$ iterations (Bubeck, 2015)

Subgradient method/“descent” - proof - I

- Iteration: $\theta_t = \Pi_D(\theta_{t-1} - \gamma_t g'(\theta_{t-1}))$ with $\gamma_t = \frac{2D}{B\sqrt{t}}$
- Assumption: $\|g'(\theta)\|_2 \leqslant B$ and $\|\theta\|_2 \leqslant D$

$$\begin{aligned}
\|\theta_t - \theta_*\|_2^2 &\leqslant \|\theta_{t-1} - \theta_* - \gamma_t g'(\theta_{t-1})\|_2^2 \text{ by contractivity of projections} \\
&\leqslant \|\theta_{t-1} - \theta_*\|_2^2 + B^2\gamma_t^2 - 2\gamma_t(\theta_{t-1} - \theta_*)^\top g'(\theta_{t-1}) \text{ because } \|g'(\theta_{t-1})\|_2 \leqslant B \\
&\leqslant \|\theta_{t-1} - \theta_*\|_2^2 + B^2\gamma_t^2 - 2\gamma_t\big[g(\theta_{t-1}) - g(\theta_*)\big] \text{ (property of subgradients)}
\end{aligned}$$

- leading to

$$g(\theta_{t-1}) - g(\theta_*) \leqslant \frac{B^2\gamma_t}{2} + \frac{1}{2\gamma_t}\big[\|\theta_{t-1} - \theta_*\|_2^2 - \|\theta_t - \theta_*\|_2^2\big]$$

Subgradient method/“descent” - proof - II

- Starting from $$g(\theta_{t-1}) - g(\theta_*) \leqslant \frac{B^2\gamma_t}{2} + \frac{1}{2\gamma_t}\big[\|\theta_{t-1}-\theta_*\|_2^2 - \|\theta_t-\theta_*\|_2^2\big]$$

- Constant step-size $\gamma_t = \gamma$

$$\begin{aligned}\sum_{u=1}^t \big[g(\theta_{u-1}) - g(\theta_*)\big] &\leqslant \sum_{u=1}^t \frac{B^2\gamma}{2} + \sum_{u=1}^t \frac{1}{2\gamma}\big[\|\theta_{u-1}-\theta_*\|_2^2 - \|\theta_u-\theta_*\|_2^2\big] \\ &\leqslant t\frac{B^2\gamma}{2} + \frac{1}{2\gamma}\|\theta_0-\theta_*\|_2^2 \leqslant t\frac{B^2\gamma}{2} + \frac{2}{\gamma}D^2\end{aligned}$$

- Optimized step-size $\gamma_t = \frac{2D}{B\sqrt{t}}$ depends on “horizon”
  - Leads to bound of $2DB\sqrt{t}$

- Using convexity: $$g\bigg(\frac{1}{t}\sum_{k=0}^{t-1}\theta_k\bigg) - g(\theta_*) \leqslant \frac{2DB}{\sqrt{t}}$$

Subgradient method/"descent" - proof - III

- Starting from $\quad g(\theta_{t-1}) - g(\theta_*) \leqslant \frac{B^2\gamma_t}{2} + \frac{1}{2\gamma_t}\big[\|\theta_{t-1}-\theta_*\|_2^2 - \|\theta_t-\theta_*\|_2^2\big]$

- Decreasing step-size

$$
\begin{aligned}
\sum_{u=1}^{t}\big[g(\theta_{u-1}) - g(\theta_*)\big] &\leqslant \sum_{u=1}^{t}\frac{B^2\gamma_u}{2} + \sum_{u=1}^{t}\frac{1}{2\gamma_u}\big[\|\theta_{u-1}-\theta_*\|_2^2 - \|\theta_u-\theta_*\|_2^2\big] \\
&= \sum_{u=1}^{t}\frac{B^2\gamma_u}{2} + \sum_{u=1}^{t-1}\|\theta_u-\theta_*\|_2^2\Big(\frac{1}{2\gamma_{u+1}} - \frac{1}{2\gamma_u}\Big) + \frac{\|\theta_0-\theta_*\|_2^2}{2\gamma_1} - \frac{\|\theta_t-\theta_*\|_2^2}{2\gamma_t} \\
&\leqslant \sum_{u=1}^{t}\frac{B^2\gamma_u}{2} + \sum_{u=1}^{t-1}4D^2\Big(\frac{1}{2\gamma_{u+1}} - \frac{1}{2\gamma_u}\Big) + \frac{4D^2}{2\gamma_1} \\
&= \sum_{u=1}^{t}\frac{B^2\gamma_u}{2} + \frac{4D^2}{2\gamma_t} \leqslant 2DB\sqrt{t} \text{ with } \gamma_t = \frac{2D}{B\sqrt{t}}
\end{aligned}
$$

- Using convexity: $\quad g\Big(\frac{1}{t}\sum_{k=0}^{t-1}\theta_k\Big) - g(\theta_*) \leqslant \frac{2DB}{\sqrt{t}}$

Subgradient descent for machine learning

- **Assumptions** (f is the expected risk, $\hat{f}$ the empirical risk)
  - "Linear" predictors: $\theta(x) = \theta^\top \Phi(x)$, with $\|\Phi(x)\|_2 \leqslant R$ a.s.
  - $\hat{f}(\theta) = \frac{1}{n}\sum_{i=1}^n \ell(y_i, \Phi(x_i)^\top \theta)$
  - G-Lipschitz loss: f and $\hat{f}$ are GR-Lipschitz on $\Theta = \{\|\theta\|_2 \leqslant D\}$

- **Statistics**: with probability greater than $1-\delta$

$$\sup_{\theta \in \Theta} |\hat{f}(\theta) - f(\theta)| \leqslant \frac{GRD}{\sqrt{n}} \left[2 + \sqrt{2 \log \frac{2}{\delta}} \right]$$

- **Optimization**: after t iterations of subgradient method

$$\hat{f}(\hat{\theta}) - \min_{\eta \in \Theta} \hat{f}(\eta) \leqslant \frac{GRD}{\sqrt{t}}$$

- $t = n$ iterations, with total running-time complexity of $O(n^2 d)$

Subgradient descent - strong convexity

- **Assumptions**
  - g convex and B-Lipschitz-continuous on $\{\|\theta\|_2 \leqslant D\}$
  - g μ-strongly convex

- **Algorithm**: $\theta_t = \Pi_D\left(\theta_{t-1} - \frac{2}{\mu(t+1)} g'(\theta_{t-1})\right)$

- **Bound**:

$$g\left(\frac{2}{t(t+1)} \sum_{k=1}^{t} k\theta_{k-1}\right) - g(\theta_*) \leqslant \frac{2B^2}{\mu(t+1)}$$

- Three-line proof

- Best possible convergence rate after $O(d)$ iterations (Bubeck, 2015)

Subgradient method - strong convexity - proof - I

- Iteration: $\theta_t = \Pi_D(\theta_{t-1} - \gamma_t g'(\theta_{t-1}))$ with $\gamma_t = \frac{2}{\mu(t+1)}$

- Assumption: $\|g'(\theta)\|_2 \leqslant B$ and $\|\theta\|_2 \leqslant D$ and μ-strong convexity of f

$$\begin{aligned}
\|\theta_t - \theta_*\|_2^2 &\leqslant \|\theta_{t-1} - \theta_* - \gamma_t g'(\theta_{t-1})\|_2^2 \text{ by contractivity of projections} \\
&\leqslant \|\theta_{t-1} - \theta_*\|_2^2 + B^2\gamma_t^2 - 2\gamma_t(\theta_{t-1} - \theta_*)^\top g'(\theta_{t-1}) \text{ because } \|g'(\theta_{t-1})\|_2 \leqslant B \\
&\leqslant \|\theta_{t-1} - \theta_*\|_2^2 + B^2\gamma_t^2 - 2\gamma_t\big[g(\theta_{t-1}) - g(\theta_*) + \frac{\mu}{2}\|\theta_{t-1} - \theta_*\|_2^2\big] \\
&\qquad \text{(property of subgradients and strong convexity)}
\end{aligned}$$

- leading to

$$\begin{aligned}
g(\theta_{t-1}) - g(\theta_*) &\leqslant \frac{B^2\gamma_t}{2} + \frac{1}{2}\Big[\frac{1}{\gamma_t} - \mu\Big]\|\theta_{t-1} - \theta_*\|_2^2 - \frac{1}{2\gamma_t}\|\theta_t - \theta_*\|_2^2 \\
&\leqslant \frac{B^2}{\mu(t+1)} + \frac{\mu}{2}\Big[\frac{t-1}{2}\Big]\|\theta_{t-1} - \theta_*\|_2^2 - \frac{\mu(t+1)}{4}\|\theta_t - \theta_*\|_2^2
\end{aligned}$$

Subgradient method - strong convexity - proof - II

- From $g(\theta_{t-1}) - g(\theta_*) \leqslant \dfrac{B^2}{\mu(t+1)} + \dfrac{\mu}{2}\big[\dfrac{t-1}{2}\big]\|\theta_{t-1} - \theta_*\|_2^2 - \dfrac{\mu(t+1)}{4}\|\theta_t - \theta_*\|_2^2$

$$\sum_{u=1}^{t} u\big[g(\theta_{u-1}) - g(\theta_*)\big] \leqslant \sum_{t=1}^{u} \frac{B^2 u}{\mu(u+1)} + \frac{1}{4}\sum_{u=1}^{t}\big[u(u-1)\|\theta_{u-1} - \theta_*\|_2^2 - u(u+1)\|\theta_u - \theta_*\|_2^2\big]$$
$$\leqslant \quad \frac{B^2 t}{\mu} + \frac{1}{4}\big[0 - t(t+1)\|\theta_t - \theta_*\|_2^2\big] \leqslant \frac{B^2 t}{\mu}$$

- Using convexity: $g\bigg(\dfrac{2}{t(t+1)}\displaystyle\sum_{u=1}^{t} u\theta_{u-1}\bigg) - g(\theta_*) \leqslant \dfrac{2B^2}{t+1}$

- NB: with step-size $\gamma_n = 1/(n\mu)$, extra logarithmic factor

Summary: minimizing convex functions

- **Assumption**: g convex

- **Gradient descent**: $\theta_t = \theta_{t-1} - \gamma_t \, g'(\theta_{t-1})$
  - $O(1/\sqrt{t})$ convergence rate for non-smooth convex functions
  - $O(1/t)$ convergence rate for smooth convex functions
  - $O(e^{-\rho t})$ convergence rate for strongly smooth convex functions

- **Newton method**: $\theta_t = \theta_{t-1} - g''(\theta_{t-1})^{-1} g'(\theta_{t-1})$
  - $O\big(e^{-\rho 2^t}\big)$ convergence rate

Summary: minimizing convex functions

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- **Newton method**: $\theta_t = \theta_{t-1} - g''(\theta_{t-1})^{-1} g'(\theta_{t-1})$
  - $O\big(e^{-\rho 2^t}\big)$ convergence rate
- **Key insights from Bottou and Bousquet (2008)**
  1. In machine learning, no need to optimize below statistical error
  2. In machine learning, cost functions are averages

$\Rightarrow$ **Stochastic approximation**

Summary of rates of convergence

- Problem parameters
  - D diameter of the domain
  - B Lipschitz-constant
  - L smoothness constant
  - μ strong convexity constant

	convex	strongly convex
nonsmooth	deterministic: $BD/\sqrt{t}$	deterministic: $B^2/(t\mu)$
smooth	deterministic: LD^2/t^2	deterministic: $\exp(-t\sqrt{\mu/L})$
quadratic	deterministic: LD^2/t^2	deterministic: $\exp(-t\sqrt{\mu/L})$

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1. **Introduction**
   - Large-scale machine learning and optimization
   - Classes of functions (convex, smooth, etc.)
   - Traditional statistical analysis through Rademacher complexity
2. **Classical methods for convex optimization**
   - Smooth optimization (gradient descent, Newton method)
   - Non-smooth optimization (subgradient descent)
3. **Classical stochastic approximation** (not covered)
   - Robbins-Monro algorithm (1951)

Outline - II

4. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds

5. **Smooth stochastic approximation algorithms**
   - Non-asymptotic analysis for smooth functions
   - Least-squares regression without decaying step-sizes

6. **Finite data sets**
   - Gradient methods with exponential convergence rates

Stochastic approximation

- **Goal**: Minimizing a function f defined on $\mathbb{R}^d$
  - given only unbiased estimates $f'_n(\theta_n)$ of its gradients $f'(\theta_n)$ at certain points $\theta_n \in \mathbb{R}^d$

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  - given only unbiased estimates $f_n'(\theta_n)$ of its gradients $f'(\theta_n)$ at certain points $\theta_n \in \mathbb{R}^d$

- **Machine learning - statistics**
  - **loss for a single pair of observations**: $\boxed{f_n(\theta) = \ell(y_n, \theta^\top \Phi(x_n))}$
  - $f(\theta) = \mathbb{E} f_n(\theta) = \mathbb{E}\, \ell(y_n, \theta^\top \Phi(x_n)) =$ **generalization error**
  - Expected gradient: $f'(\theta) = \mathbb{E} f_n'(\theta) = \mathbb{E}\left\{\ell'(y_n, \theta^\top \Phi(x_n))\, \Phi(x_n)\right\}$
  - Non-asymptotic results

- **Number of iterations = number of observations**

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- **Stochastic approximation**
  - (much) broader applicability beyond convex optimization

$$\theta_n = \theta_{n-1} - \gamma_n h_n(\theta_{n-1}) \text{ with } \mathbb{E}\big[h_n(\theta_{n-1})|\theta_{n-1}\big] = h(\theta_{n-1})$$

  - Beyond convex problems, i.i.d assumption, finite dimension, etc.
  - Typically asymptotic results
  - See, e.g., Kushner and Yin (2003); Benveniste et al. (2012)

Relationship to online learning

- **Stochastic approximation**
  - Minimize $f(\theta) = \mathbb{E}_z \ell(\theta, z)$ = **generalization error** of θ
  - Using the gradients of single i.i.d. observations

Relationship to online learning

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- **Batch learning**
  - Finite set of observations: $z_1, \dots, z_n$
  - Empirical risk: $\hat{f}(\theta) = \frac{1}{n} \sum_{k=1}^{n} \ell(\theta, z_i)$
  - Estimator $\hat{\theta}$ = Minimizer of $\hat{f}(\theta)$ over a certain class Θ
  - Generalization bound using uniform concentration results

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- **Online learning**
  - Update $\hat{\theta}_n$ after each new (potentially adversarial) observation z_n
  - Cumulative loss: $\frac{1}{n} \sum_{k=1}^{n} \ell(\hat{\theta}_{k-1}, z_k)$
  - Online to batch through averaging (Cesa-Bianchi et al., 2004)

Convex stochastic approximation

- **Key properties of f and/or f_n**
  - Smoothness: f B-Lipschitz continuous, f' L-Lipschitz continuous
  - Strong convexity: f μ-strongly convex

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$$\boxed{\theta_n = \theta_{n-1} - \gamma_n f_n'(\theta_{n-1})}$$

  - Polyak-Ruppert averaging: $\bar{\theta}_n = \frac{1}{n} \sum_{k=0}^{n-1} \theta_k$
  - Which learning rate sequence γ_n? Classical setting: $\boxed{\gamma_n = C n^{-\alpha}}$

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- **Desirable practical behavior**
  - Applicable (at least) to classical supervised learning problems
  - Robustness to (potentially unknown) constants (L,B,μ)
  - Adaptivity to difficulty of the problem (e.g., strong convexity)

Stochastic subgradient "descent"/method

- **Assumptions**
  - f_n convex and B-Lipschitz-continuous on $\{\|\theta\|_2 \leqslant D\}$
  - (f_n) i.i.d. functions such that $\mathbb{E} f_n = f$
  - θ_* global optimum of f on $\mathcal{C} = \{\|\theta\|_2 \leqslant D\}$

- **Algorithm**: $\theta_n = \Pi_D\bigg(\theta_{n-1} - \dfrac{2D}{B\sqrt{n}} f_n'(\theta_{n-1})\bigg)$

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- **Bound**:

$$\mathbb{E} f\bigg(\frac{1}{n}\sum_{k=0}^{n-1}\theta_k\bigg) - f(\theta_*) \leqslant \frac{2DB}{\sqrt{n}}$$

- "Same" three-line proof as in the deterministic case
- Minimax rate (Nemirovsky and Yudin, 1983; Agarwal et al., 2012)
- Running-time complexity: $O(dn)$ after n iterations

Stochastic subgradient method - proof - I

- Iteration: $\theta_n = \Pi_D(\theta_{n-1} - \gamma_n f_n'(\theta_{n-1}))$ with $\gamma_n = \frac{2D}{B\sqrt{n}}$

- $\mathcal{F}_n$: information up to time n

- $\|f_n'(\theta)\|_2 \leqslant B$ and $\|\theta\|_2 \leqslant D$, unbiased gradients/functions $\mathbb{E}(f_n|\mathcal{F}_{n-1}) = f$

$$
\begin{aligned}
\|\theta_n - \theta_*\|_2^2 &\leqslant \|\theta_{n-1} - \theta_* - \gamma_n f_n'(\theta_{n-1})\|_2^2 \text{ by contractivity of projections} \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f_n'(\theta_{n-1}) \text{ because } \|f_n'(\theta_{n-1})\|_2 \leqslant B
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\big[\|\theta_n - \theta_*\|_2^2 | \mathcal{F}_{n-1}\big] &\leqslant \|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f'(\theta_{n-1}) \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n\big[f(\theta_{n-1}) - f(\theta_*)\big] \text{ (subgradient property)} \\
\mathbb{E}\|\theta_n - \theta_*\|_2^2 &\leqslant \mathbb{E}\|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n\big[\mathbb{E}f(\theta_{n-1}) - f(\theta_*)\big]
\end{aligned}
$$

- leading to $\mathbb{E}f(\theta_{n-1}) - f(\theta_*) \leqslant \dfrac{B^2\gamma_n}{2} + \dfrac{1}{2\gamma_n}\big[\mathbb{E}\|\theta_{n-1} - \theta_*\|_2^2 - \mathbb{E}\|\theta_n - \theta_*\|_2^2\big]$

Stochastic subgradient method - proof - II

- Starting from $\mathbb{E}f(\theta_{n-1}) - f(\theta_*) \leqslant \frac{B^2\gamma_n}{2} + \frac{1}{2\gamma_n}\big[\mathbb{E}\|\theta_{n-1}-\theta_*\|_2^2 - \mathbb{E}\|\theta_n-\theta_*\|_2^2\big]$

$$\begin{aligned}\sum_{u=1}^{n}\big[\mathbb{E}f(\theta_{u-1}) - f(\theta_*)\big] &\leqslant \sum_{u=1}^{n}\frac{B^2\gamma_u}{2} + \sum_{u=1}^{n}\frac{1}{2\gamma_u}\big[\mathbb{E}\|\theta_{u-1}-\theta_*\|_2^2 - \mathbb{E}\|\theta_u-\theta_*\|_2^2\big] \\ &\leqslant \sum_{u=1}^{n}\frac{B^2\gamma_u}{2} + \frac{4D^2}{2\gamma_n} \leqslant 2DB\sqrt{n} \text{ with } \gamma_n = \frac{2D}{B\sqrt{n}}\end{aligned}$$

- Using convexity: $\mathbb{E}f\left(\frac{1}{n}\sum_{k=0}^{n-1}\theta_k\right) - f(\theta_*) \leqslant \frac{2DB}{\sqrt{n}}$

Stochastic subgradient descent - strong convexity - I

- **Assumptions**
  - f_n convex and B-Lipschitz-continuous
  - (f_n) i.i.d. functions such that $\mathbb{E}f_n = f$
  - f μ-strongly convex on $\{\|\theta\|_2 \leqslant D\}$
  - θ_* global optimum of f over $\{\|\theta\|_2 \leqslant D\}$

- **Algorithm**: $\theta_n = \Pi_D\bigg(\theta_{n-1} - \dfrac{2}{\mu(n+1)} f_n'(\theta_{n-1})\bigg)$

- **Bound**:

$$\mathbb{E}f\bigg(\frac{2}{n(n+1)}\sum_{k=1}^n k\theta_{k-1}\bigg) - f(\theta_*) \leqslant \frac{2B^2}{\mu(n+1)}$$

- "Same" proof than deterministic case (Lacoste-Julien et al., 2012)

- Minimax rate (Nemirovsky and Yudin, 1983; Agarwal et al., 2012)

Stochastic subgradient - strong convexity - proof - I

- Iteration: $\theta_n = \Pi_D(\theta_{n-1} - \gamma_n f_n'(\theta_{t-1}))$ with $\gamma_n = \frac{2}{\mu(n+1)}$

- Assumption: $\|f_n'(\theta)\|_2 \leqslant B$ and $\|\theta\|_2 \leqslant D$ and μ-strong convexity of f

$$\begin{aligned}
\|\theta_n - \theta_*\|_2^2 &\leqslant \|\theta_{n-1} - \theta_* - \gamma_n f_n'(\theta_{t-1})\|_2^2 \text{ by contractivity of projections} \\
&\leqslant \|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n(\theta_{n-1} - \theta_*)^\top f_n'(\theta_{t-1}) \text{ because } \|f_n'(\theta_{t-1})\|_2 \leqslant B \\
\mathbb{E}(\cdot|\mathcal{F}_{n-1}) &\leqslant \|\theta_{n-1} - \theta_*\|_2^2 + B^2\gamma_n^2 - 2\gamma_n\big[f(\theta_{n-1}) - f(\theta_*) + \frac{\mu}{2}\|\theta_{n-1} - \theta_*\|_2^2\big]
\end{aligned}$$

(property of subgradients and strong convexity)

- leading to

$$\begin{aligned}
\mathbb{E}f(\theta_{n-1}) - f(\theta_*) &\leqslant \frac{B^2\gamma_n}{2} + \frac{1}{2}\Big[\frac{1}{\gamma_n} - \mu\Big]\|\theta_{n-1} - \theta_*\|_2^2 - \frac{1}{2\gamma_n}\|\theta_n - \theta_*\|_2^2 \\
&\leqslant \frac{B^2}{\mu(n+1)} + \frac{\mu}{2}\Big[\frac{n-1}{2}\Big]\|\theta_{n-1} - \theta_*\|_2^2 - \frac{\mu(n+1)}{4}\|\theta_n - \theta_*\|_2^2
\end{aligned}$$

Stochastic subgradient - strong convexity - proof - II

- From $\mathbb{E} f(\theta_{n-1}) - f(\theta_*) \leqslant \frac{B^2}{\mu(n+1)} + \frac{\mu}{2}\left[\frac{n-1}{2}\right]\mathbb{E}\|\theta_{n-1}-\theta_*\|_2^2 - \frac{\mu(n+1)}{4}\mathbb{E}\|\theta_n - \theta_*\|_2^2$

$$\sum_{u=1}^{n} u\big[\mathbb{E} f(\theta_{u-1}) - f(\theta_*)\big] \leqslant \sum_{u=1}^{n} \frac{B^2 u}{\mu(u+1)} + \frac{1}{4}\sum_{u=1}^{n}\big[u(u-1)\mathbb{E}\|\theta_{u-1}-\theta_*\|_2^2 - u(u+1)\mathbb{E}\|\theta_u - \theta_*$$

$$\leqslant \quad \frac{B^2 n}{\mu} + \frac{1}{4}\big[0 - n(n+1)\mathbb{E}\|\theta_n - \theta_*\|_2^2\big] \leqslant \frac{B^2 n}{\mu}$$

- Using convexity: $\mathbb{E} f\left(\frac{2}{n(n+1)}\sum_{u=1}^{n} u\theta_{u-1}\right) - g(\theta_*) \leqslant \frac{2B^2}{n+1}$

- NB: with step-size $\gamma_n = 1/(n\mu)$, extra logarithmic factor (see later)

Stochastic subgradient descent - strong convexity - II

- **Assumptions**
  - f_n convex and B-Lipschitz-continuous
  - (f_n) i.i.d. functions such that $\mathbb{E} f_n = f$
  - θ_* global optimum of $g = f + \frac{\mu}{2}\| \cdot \|_2^2$
  - No compactness assumption - no projections
- **Algorithm**:

$$\theta_n = \theta_{n-1} - \frac{2}{\mu(n+1)} g_n'(\theta_{n-1}) = \theta_{n-1} - \frac{2}{\mu(n+1)} \big[f_n'(\theta_{n-1}) + \mu \theta_{n-1} \big]$$

- **Bound**: $\mathbb{E} g \left(\frac{2}{n(n+1)} \sum_{k=1}^{n} k \theta_{k-1} \right) - g(\theta_*) \leqslant \frac{2B^2}{\mu(n+1)}$
- Minimax convergence rate

Beyond convergence in expectation

- **Typical result**: $\mathbb{E} f\left(\frac{1}{n}\sum_{k=0}^{n-1}\theta_k\right) - f(\theta_*) \leqslant \frac{2DB}{\sqrt{n}}$
  - Obtained with simple conditioning arguments

- **High-probability bounds**
  - Markov inequality: $\mathbb{P}\left(f\left(\frac{1}{n}\sum_{k=0}^{n-1}\theta_k\right) - f(\theta_*) \geqslant \varepsilon\right) \leqslant \frac{2DB}{\sqrt{n}\varepsilon}$

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  - Concentration inequality (Nemirovski et al., 2009; Nesterov and Vial, 2008)

$$\mathbb{P}\left(f\left(\frac{1}{n}\sum_{k=0}^{n-1}\theta_k\right) - f(\theta_*) \geqslant \frac{2DB}{\sqrt{n}}(2+4t)\right) \leqslant 2\exp(-t^2)$$

- See also Bach (2013) for logistic regression

Beyond stochastic gradient method

- **Adding a proximal step**
  - Goal: $\min_{\theta \in \mathbb{R}^d} f(\theta) + \Omega(\theta) = \mathbb{E} f_n(\theta) + \Omega(\theta)$
  - Replace recursion $\theta_n = \theta_{n-1} - \gamma_n f_n'(\theta_n)$ by

$$\theta_n = \min_{\theta \in \mathbb{R}^d} \big\| \theta - \theta_{n-1} + \gamma_n f_n'(\theta_n) \big\|_2^2 + C\Omega(\theta)$$

  - Xiao (2010); Hu et al. (2009)
  - May be accelerated (Ghadimi and Lan, 2013)

- **Related frameworks**
  - Regularized dual averaging (Nesterov, 2009; Xiao, 2010)
  - Mirror descent (Nemirovski et al., 2009; Lan et al., 2012)

Minimax rates (Agarwal et al., 2012)

- **Model of computation (i.e., algorithms): first-order oracle**
  - Queries a function f by obtaining $f(\theta_k)$ and $f'(\theta_k)$ with zero-mean bounded variance noise, for $k = 0, \dots, n-1$ and outputs θ_n

- **Class of functions**
  - convex B-Lipschitz-continuous (w.r.t. ℓ_2-norm) on a compact convex set $\mathcal{C}$ containing an ℓ_∞-ball

- **Performance measure**
  - for a given algorithm and function $\varepsilon_n(\text{algo}, f) = f(\theta_n) - \inf_{\theta \in \mathcal{C}} f(\theta)$
  - for a given algorithm: $\displaystyle \sup_{\text{functions } f} \varepsilon_n(\text{algo}, f)$

- **Minimax performance**: $\displaystyle \inf_{\text{algo}} \sup_{\text{functions } f} \varepsilon_n(\text{algo}, f)$

Minimax rates (Agarwal et al., 2012)

- **Convex functions**: domain $\mathcal{C}$ that contains an ℓ_∞-ball of radius D

$$\inf_{\text{algo}} \sup_{\text{functions } f} \varepsilon(\text{algo}, f) \geqslant \text{ cst } \times \min\left\{BD\sqrt{\frac{d}{n}}, BD\right\}$$

  - Consequences for ℓ_2-ball of radius D: $BD/\sqrt{n}$
  - Upper-bound through stochastic subgradient

- μ-**strongly-convex functions**:

$$\inf_{\text{algo}} \sup_{\text{functions } f} \varepsilon_n(\text{algo}, f) \geqslant \text{ cst } \times \min\left\{\frac{B^2}{\mu n}, \frac{B^2}{\mu d}, BD\sqrt{\frac{d}{n}}, BD\right\}$$

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Convex stochastic approximation
Existing work

- **Known global minimax rates of convergence for non-smooth problems** (Nemirovsky and Yudin, 1983; Agarwal et al., 2012)
  - Strongly convex: $O((\mu n)^{-1})$
    Attained by averaged stochastic gradient descent with $\gamma_n \propto (\mu n)^{-1}$
  - Non-strongly convex: $O(n^{-1/2})$
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- **Many contributions in optimization and online learning:** Bottou and Le Cun (2005); Bottou and Bousquet (2008); Hazan et al. (2007); Shalev-Shwartz and Srebro (2008); Shalev-Shwartz et al. (2007, 2009); Xiao (2010); Duchi and Singer (2009); Nesterov and Vial (2008); Nemirovski et al. (2009)

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- **Asymptotic analysis of averaging** (Polyak and Juditsky, 1992; Ruppert, 1988)
  - All step sizes $\gamma_n = Cn^{-\alpha}$ with $\alpha \in (1/2, 1)$ lead to $O(n^{-1})$ for smooth strongly convex problems

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- **Non-asymptotic analysis for smooth problems?**

Smoothness/convexity assumptions

- Iteration: $\boxed{\theta_n = \theta_{n-1} - \gamma_n f_n'(\theta_{n-1})}$
  - Polyak-Ruppert averaging: $\bar{\theta}_n = \frac{1}{n}\sum_{k=0}^{n-1}\theta_k$
- **Smoothness of f_n**: For each $n \geqslant 1$, the function f_n is a.s. convex, differentiable with L-Lipschitz-continuous gradient f_n':
  - Smooth loss and bounded data
- **Strong convexity of f**: The function f is strongly convex with respect to the norm $\|\cdot\|$, with convexity constant $\mu > 0$:
  - Invertible population covariance matrix
  - or regularization by $\frac{\mu}{2}\|\theta\|^2$

Summary of new results (Bach and Moulines, 2011)

- Stochastic gradient descent with learning rate $\gamma_n = Cn^{-\alpha}$
- **Strongly convex smooth objective functions**
  - Old: $O(n^{-1})$ rate achieved without averaging for $\alpha = 1$
  - New: $O(n^{-1})$ rate achieved with averaging for $\alpha \in [1/2, 1]$
  - Non-asymptotic analysis with explicit constants
  - Forgetting of initial conditions
  - Robustness to the choice of C

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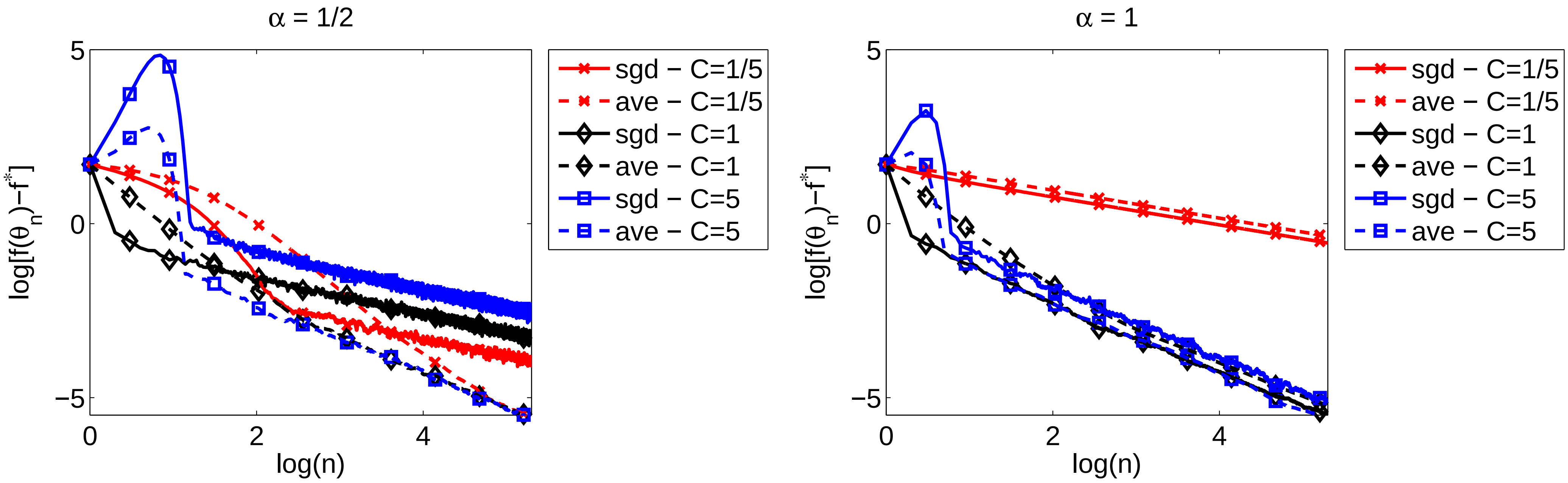
- **Convergence rates** for $\mathbb{E}\|\theta_n - \theta_*\|^2$ and $\mathbb{E}\|\bar{\theta}_n - \theta_*\|^2$
  - no averaging: $O\Big(\frac{\sigma^2\gamma_n}{\mu}\Big) + O(e^{-\mu n\gamma_n})\|\theta_0 - \theta_*\|^2$
  - averaging: $\frac{\operatorname{tr} H(\theta_*)^{-1}}{n} + \mu^{-1}O(n^{-2\alpha}+n^{-2+\alpha}) + O\Big(\frac{\|\theta_0 - \theta_*\|^2}{\mu^2 n^2}\Big)$

Robustness to wrong constants for $\gamma_n = Cn^{-\alpha}$

- $f(\theta) = \frac{1}{2}|\theta|^2$ with i.i.d. Gaussian noise ($d = 1$)
- Left: $\alpha = 1/2$
- Right: $\alpha = 1$



- See also `http://leon.bottou.org/projects/sgd`

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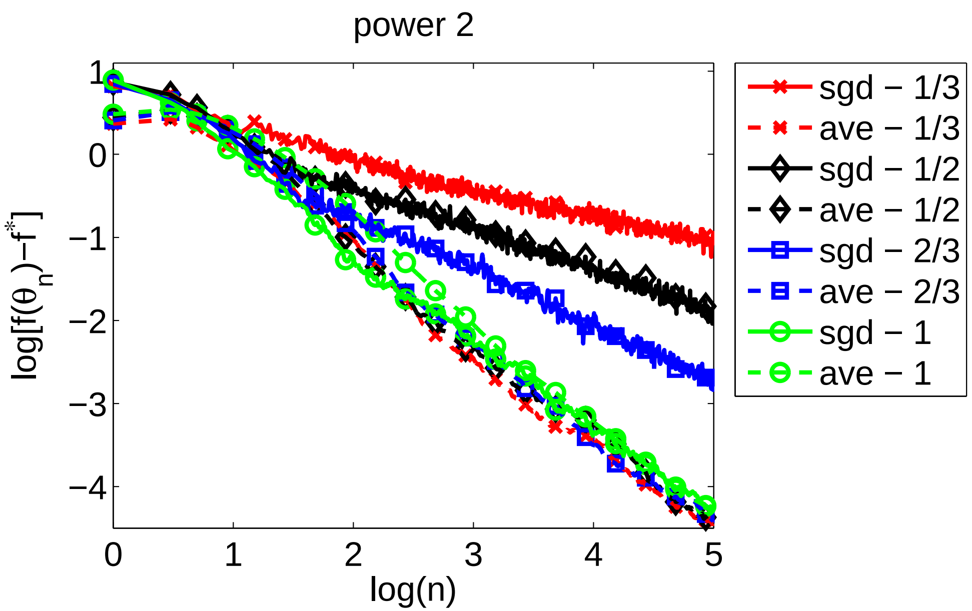
  - Non-asymptotic analysis with explicit constants

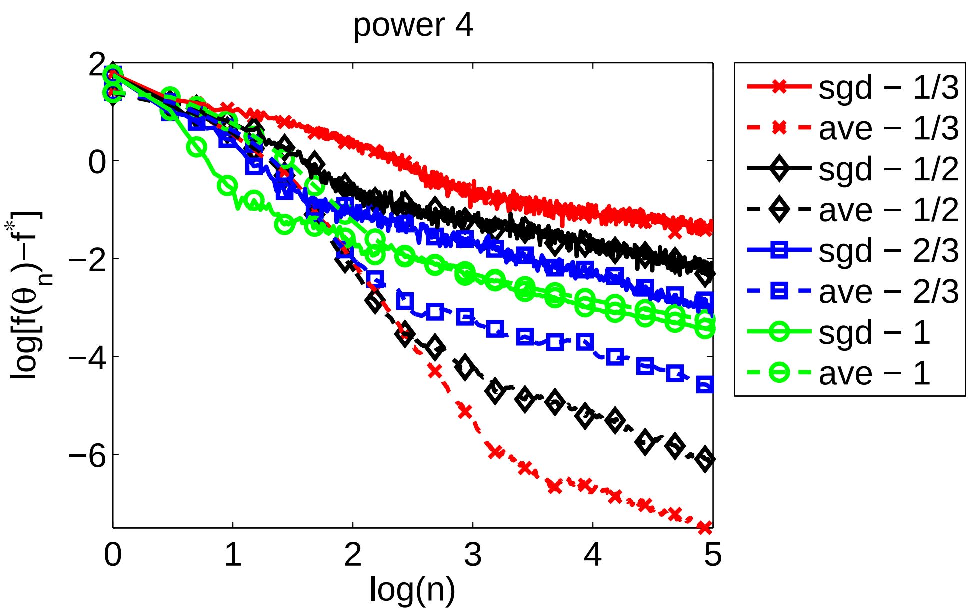
Summary of new results (Bach and Moulines, 2011)

- Stochastic gradient descent with learning rate $\gamma_n = Cn^{-\alpha}$
- **Strongly convex smooth objective functions**
  - Old: $O(n^{-1})$ rate achieved without averaging for $\alpha = 1$
  - New: $O(n^{-1})$ rate achieved with averaging for $\alpha \in [1/2, 1]$
  - Non-asymptotic analysis with explicit constants
- **Non-strongly convex smooth objective functions**
  - Old: $O(n^{-1/2})$ rate achieved with averaging for $\alpha = 1/2$
  - New: $O(\max\{n^{1/2-3\alpha/2}, n^{-\alpha/2}, n^{\alpha-1}\})$ rate achieved without averaging for $\alpha \in [1/3, 1]$
- **Take-home message**
  - Use $\alpha = 1/2$ with averaging to be adaptive to strong convexity

Robustness to lack of strong convexity

- Left: $f(\theta) = |\theta|^2$ between -1 and 1
- Right: $f(\theta) = |\theta|^4$ between -1 and 1
- affine outside of $[-1, 1]$, continuously differentiable.





Convex stochastic approximation
Existing work

- **Known global minimax rates of convergence for non-smooth problems** (Nemirovsky and Yudin, 1983; Agarwal et al., 2012)
  - Strongly convex: $O((\mu n)^{-1})$
    Attained by averaged stochastic gradient descent with $\gamma_n \propto (\mu n)^{-1}$
  - Non-strongly convex: $O(n^{-1/2})$
    Attained by averaged stochastic gradient descent with $\gamma_n \propto n^{-1/2}$

- **Asymptotic analysis of averaging** (Polyak and Juditsky, 1992; Ruppert, 1988)
  - All step sizes $\gamma_n = Cn^{-\alpha}$ with $\alpha \in (1/2, 1)$ lead to $O(n^{-1})$ for smooth strongly convex problems

- **A single adaptive algorithm for smooth problems with convergence rate $O(\min\{1/\mu n, 1/\sqrt{n}\})$ in all situations?**

Adaptive algorithm for logistic regression

- **Logistic regression**: $(\Phi(x_n), y_n) \in \mathbb{R}^d \times \{-1, 1\}$
  - Single data point: $f_n(\theta) = \log(1 + \exp(-y_n \theta^\top \Phi(x_n)))$
  - Generalization error: $f(\theta) = \mathbb{E} f_n(\theta)$

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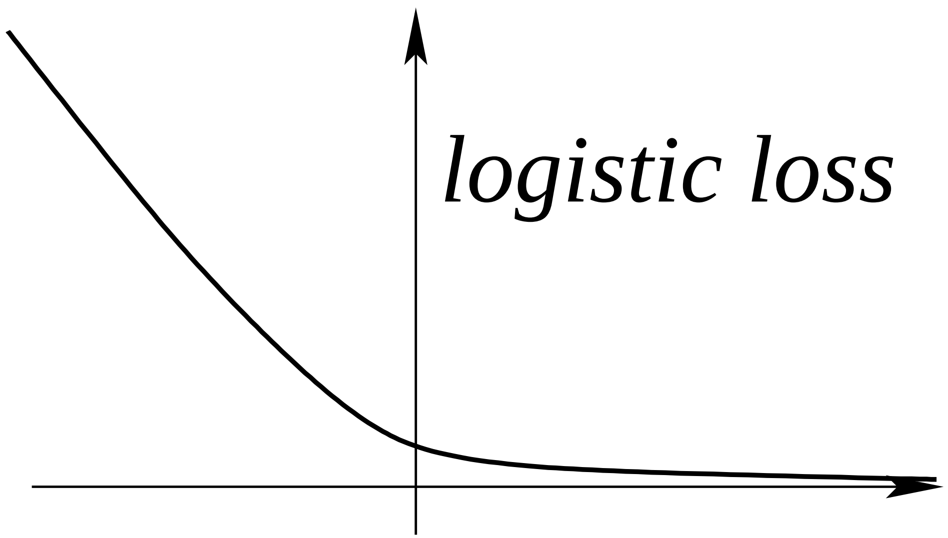
- **Cannot be strongly convex** $\Rightarrow$ local strong convexity
  - unless restricted to $|\theta^\top \Phi(x_n)| \leqslant M$ (with constants e^M - *proof*)
  - $\mu =$ lowest eigenvalue of the Hessian at the optimum $f''(\theta_*)$



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- n **steps of averaged SGD with constant step-size** $1/\big(2R^2\sqrt{n}\big)$
  - with $R =$ radius of data (Bach, 2013):

$$\mathbb{E} f(\bar{\theta}_n) - f(\theta_*) \leqslant \min\left\{\frac{1}{\sqrt{n}}, \frac{R^2}{n\mu}\right\}\big(15 + 5R\|\theta_0 - \theta_*\|\big)^4$$

  - Proof based on self-concordance (Nesterov and Nemirovski, 1994)

Self-concordance

- Usual definition for convex $\varphi : \mathbb{R} \to \mathbb{R}$: $|\varphi'''(t)| \leqslant 2\varphi''(t)^{3/2}$
  - Affine invariant
  - Extendable to all convex functions on $\mathbb{R}^d$ by looking at rays
  - Used for the sharp proof of quadratic convergence of Newton method (Nesterov and Nemirovski, 1994)

- Generalized notion: $|\varphi'''(t)| \leqslant \varphi''(t)$
  - Applicable to logistic regression (with extensions)
  - $\varphi(t) = \log(1 + e^{-t})$, $\varphi'(t) = (1 + e^{t})^{-1}$, etc...

- Important properties
  - Allows global Taylor expansions
  - Relates expansions of derivatives of different orders

Adaptive algorithm for logistic regression

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Least-mean-square algorithm

- **Least-squares**: $f(\theta) = \frac{1}{2}\mathbb{E}\big[(y_n - \langle \Phi(x_n), \theta \rangle)^2\big]$ with $\theta \in \mathbb{R}^d$
  - SGD = least-mean-square algorithm (see, e.g., Macchi, 1995)
  - usually studied without averaging and decreasing step-sizes
  - with strong convexity assumption $\mathbb{E}\big[\Phi(x_n) \otimes \Phi(x_n)\big] = H \succcurlyeq \mu \cdot \mathrm{Id}$

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- **New analysis for averaging and constant step-size** $\gamma = 1/(4R^2)$
  - Assume $\|\Phi(x_n)\| \leqslant R$ and $|y_n - \langle \Phi(x_n), \theta_*\rangle| \leqslant \sigma$ almost surely
  - No assumption regarding lowest eigenvalues of H
  - Main result: $\boxed{\mathbb{E} f(\bar{\theta}_{n-1}) - f(\theta_*) \leqslant \frac{4\sigma^2 d}{n} + \frac{4R^2\|\theta_0 - \theta_*\|^2}{n}}$
- **Matches statistical lower bound** (Tsybakov, 2003)
  - Non-asymptotic robust version of Györfi and Walk (1996)

Least-squares - Proof technique - I

- LMS recursion:

$$\theta_n - \theta_* = \big[I - \gamma \Phi(x_n) \otimes \Phi(x_n)\big](\theta_{n-1} - \theta_*) + \gamma\, \varepsilon_n \Phi(x_n)$$

- Simplified LMS recursion: with $H = \mathbb{E}\big[\Phi(x_n) \otimes \Phi(x_n)\big]$

$$\theta_n - \theta_* = \big[I - \gamma H\big](\theta_{n-1} - \theta_*) + \gamma\, \varepsilon_n \Phi(x_n)$$

  - Direct proof technique of Polyak and Juditsky (1992), e.g.,

$$\theta_n - \theta_* = \big[I - \gamma H\big]^n(\theta_0 - \theta_*) + \gamma \sum_{k=1}^{n} \big[I - \gamma H\big]^{n-k} \varepsilon_k \Phi(x_k)$$

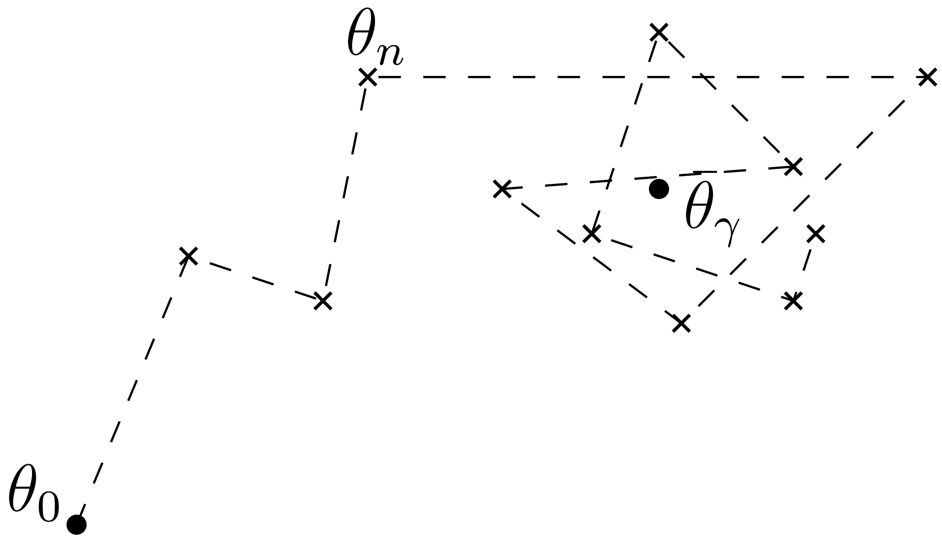
- Infinite expansion of Aguech, Moulines, and Priouret (2000) in powers of γ

Markov chain interpretation of constant step sizes

- LMS recursion for $f_n(\theta) = \frac{1}{2}\big(y_n - \langle \Phi(x_n), \theta \rangle\big)^2$

$$\theta_n = \theta_{n-1} - \gamma\big(\langle \Phi(x_n), \theta_{n-1} \rangle - y_n\big)\Phi(x_n)$$

- The sequence $(\theta_n)_n$ is a homogeneous Markov chain
  - convergence to a stationary distribution π_γ
  - with expectation $\bar{\theta}_\gamma \stackrel{\text{def}}{=} \int \theta \pi_\gamma(\mathrm{d}\theta)$



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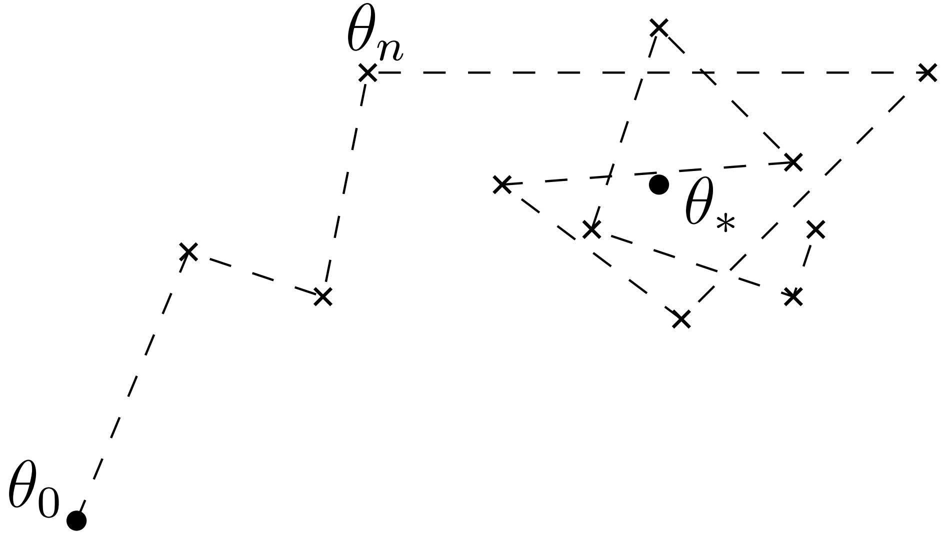

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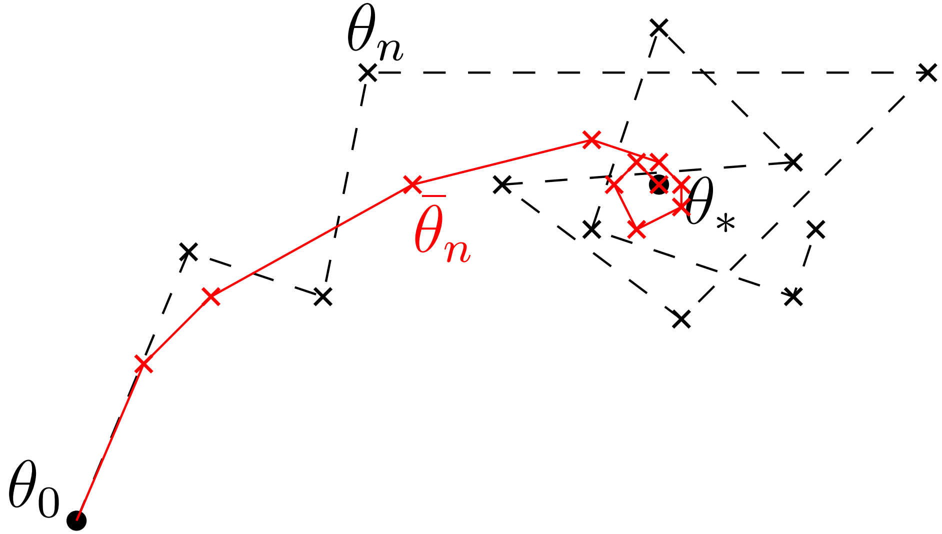

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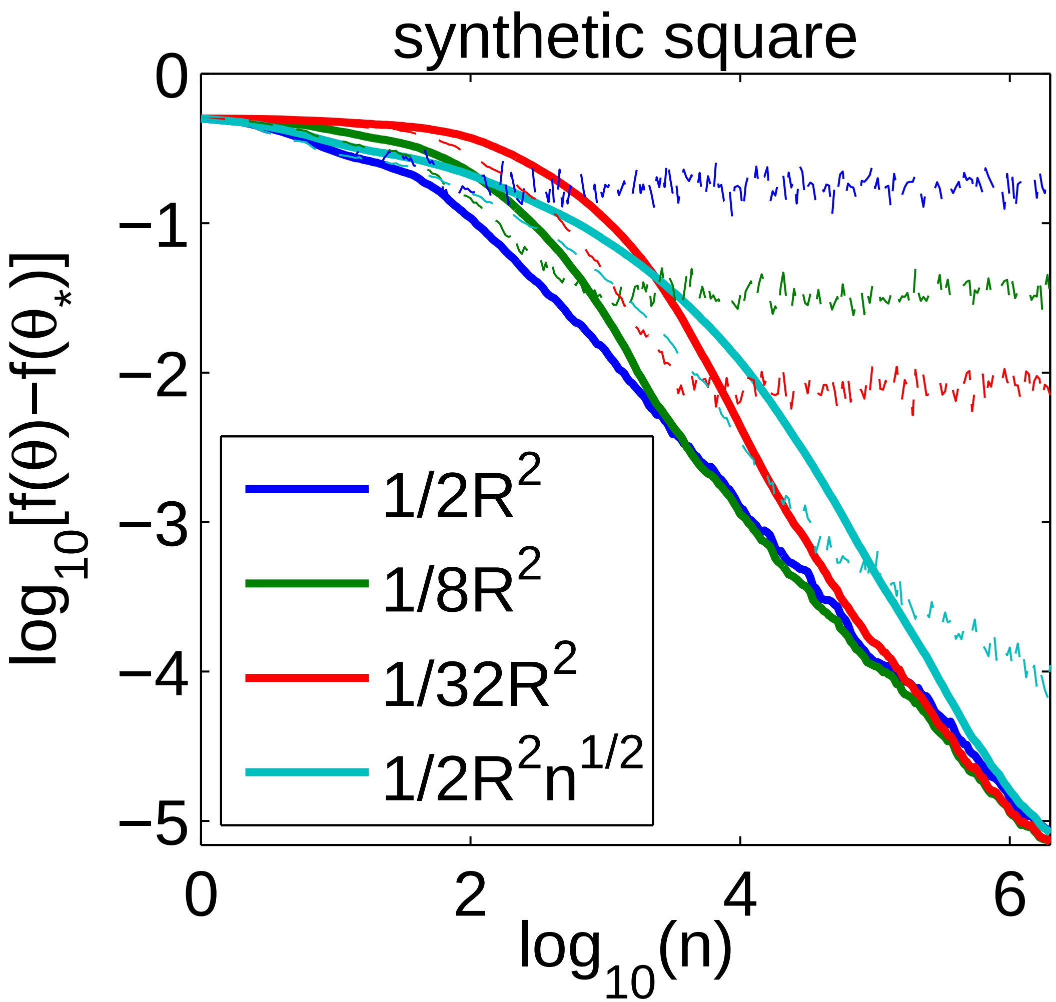
- **For least-squares, $\bar{\theta}_\gamma = \theta_*$**
  - θ_n does not converge to θ_* but oscillates around it
  - oscillations of order $\sqrt{\gamma}$
- **Ergodic theorem:**
  - Averaged iterates converge to $\bar{\theta}_\gamma = \theta_*$ at rate $O(1/n)$

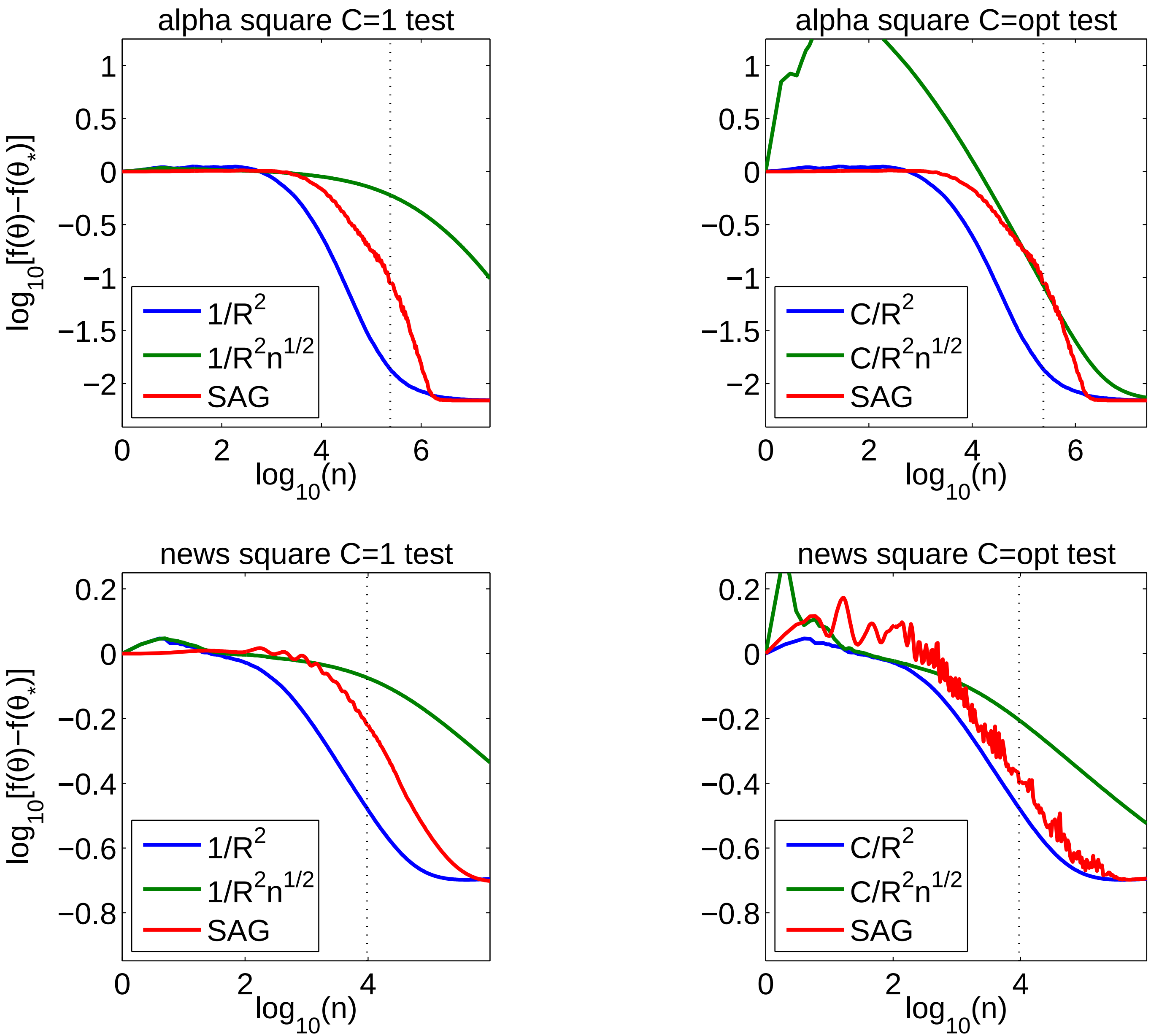
Simulations - synthetic examples

- Gaussian distributions - $d = 20$



Simulations - benchmarks

- *alpha* $(d = 500,\ n = 500\ 000)$, *news* $(d = 1\ 300\ 000,\ n = 20\ 000)$



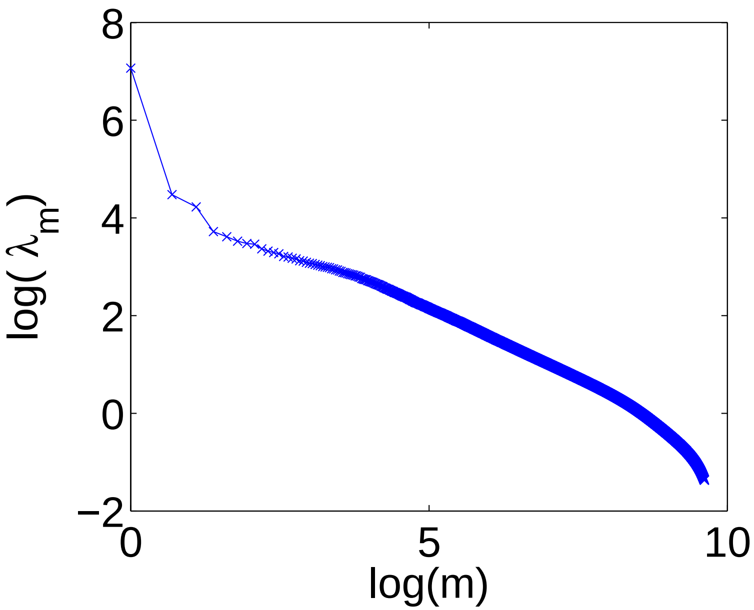
Optimal bounds for least-squares?

- **Least-squares**: cannot beat $\sigma^2 d/n$ (Tsybakov, 2003). Really?
  - What if $d \gg n$?

- **Refined assumptions with adaptivity** (Dieuleveut and Bach, 2014)
  - Beyond strong convexity or lack thereof

Finer assumptions (Dieuleveut and Bach, 2014)

- **Covariance eigenvalues**
  - Pessimistic assumption: all eigenvalues λ_m less than a constant
  - Actual decay as $\lambda_m = o(m^{-\alpha})$ with $\operatorname{tr} H^{1/\alpha} = \sum_m \lambda_m^{1/\alpha}$ small



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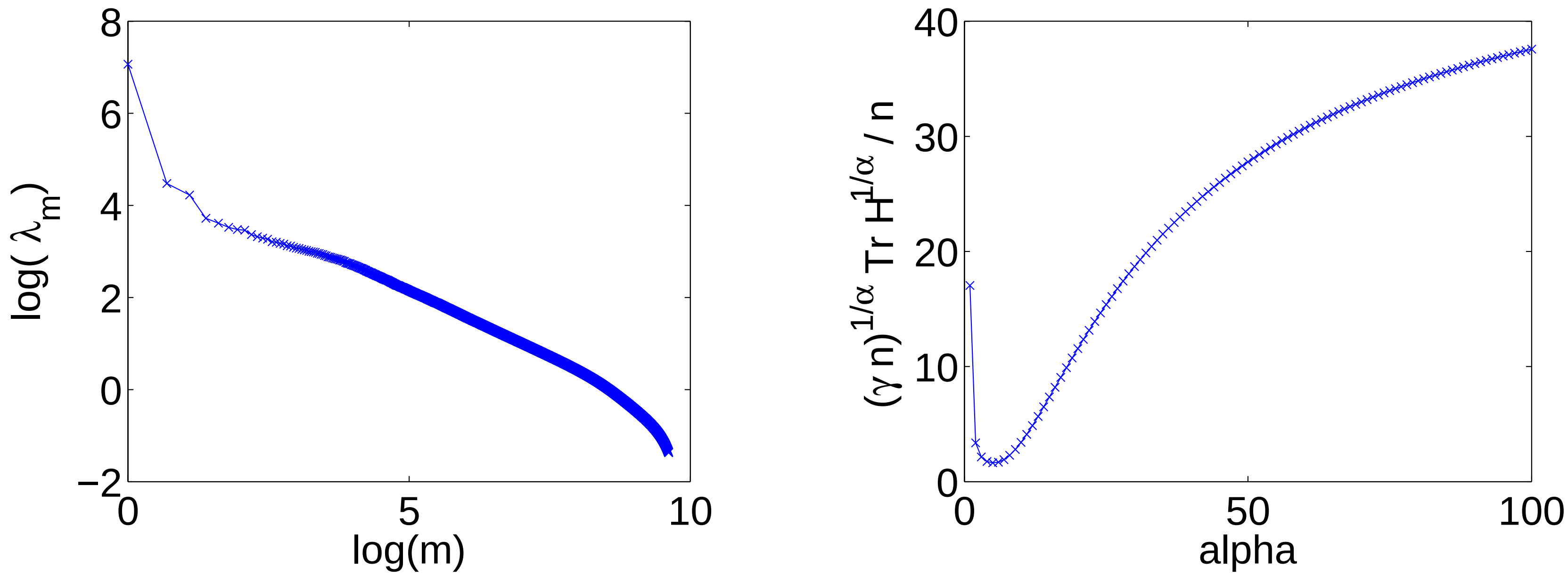
  - New result: replace $\dfrac{\sigma^2 d}{n}$ by $\dfrac{\sigma^2 (\gamma n)^{1/\alpha} \operatorname{tr} H^{1/\alpha}}{n}$



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- **Optimal predictor**
  - Pessimistic assumption: $\|\theta_0 - \theta_*\|^2$ finite
  - Finer assumption: $\|H^{1/2-r}(\theta_0 - \theta_*)\|_2$ small
  - Replace $\dfrac{\|\theta_0 - \theta_*\|^2}{\gamma n}$ by $\dfrac{4\|H^{1/2-r}(\theta_0 - \theta_*)\|_2}{\gamma^{2r} n^{2\min\{r,1\}}}$

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$$f(\bar{\theta}_n) - f(\theta_*) \leqslant \frac{16\sigma^2 \operatorname{tr} H^{1/\alpha}}{n}(\gamma n)^{1/\alpha} + \frac{4\|H^{1/2-r}(\theta_0 - \theta_*)\|_2}{\gamma^{2r} n^{2\min\{r,1\}}}$$

  - Previous results: $\alpha = +\infty$ and $r = 1/2$
  - Valid for all α and r
  - Optimal step-size potentially decaying with n
  - Extension to non-parametric estimation (kernels) with optimal rates

From least-squares to non-parametric estimation - I

- **Extension to Hilbert spaces**: $\Phi(x), \theta \in \mathcal{H}$

$$\theta_n = \theta_{n-1} - \gamma\big(\langle \Phi(x_n), \theta_{n-1}\rangle - y_n\big)\Phi(x_n)$$

- **If $\theta_0 = 0$, θ_n is a linear combination of** $\Phi(x_1), \dots, \Phi(x_n)$

$$\theta_n = \sum_{k=1}^{n} \alpha_k \Phi(x_k) \quad \text{and} \quad \alpha_n = -\gamma \sum_{k=1}^{n-1} \alpha_k \langle \Phi(x_k), \Phi(x_n)\rangle + \gamma y_n$$

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- **Kernel trick**: $k(x, x') = \langle \Phi(x), \Phi(x')\rangle$
  - Reproducing kernel Hilbert spaces and non-parametric estimation
  - See, e.g., Schölkopf and Smola (2001); Shawe-Taylor and Cristianini (2004); Dieuleveut and Bach (2014)
  - Still $O(n^2)$

From least-squares to non-parametric estimation - II

- **Simple example**: Sobolev space on $\mathcal{X} = [0, 1]$
  - $\Phi(x) =$ weighted Fourier basis $\Phi(x)_j = \varphi_j \cos(2j\pi x)$ (plus sine)
  - kernel $k(x, x') = \sum_j \varphi_j^2 \cos\big[2j\pi(x - x')\big]$
  - Optimal prediction function θ_* has norm $\|\theta_*\|^2 = \sum_j |\mathcal{F}(\theta_*)_j|^2 \varphi_j^{-2}$
  - Depending on smoothness, may or may not be finite

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- Adapted norm $\|H^{1/2-r}\theta_*\|^2 = \sum_j |\mathcal{F}(\theta_*)_j|^2 \varphi_j^{-4r}$ may be finite

$$f(\bar{\theta}_n) - f(\theta_*) \leqslant \frac{16\sigma^2 \operatorname{tr} H^{1/\alpha}}{n}(\gamma n)^{1/\alpha} + \frac{4\|H^{1/2-r}(\theta_0 - \theta_*)\|_2}{\gamma^{2r} n^{2\min\{r,1\}}}$$

- **Same effect than ℓ_2-regularization with weight λ equal to** $\dfrac{1}{\gamma n}$

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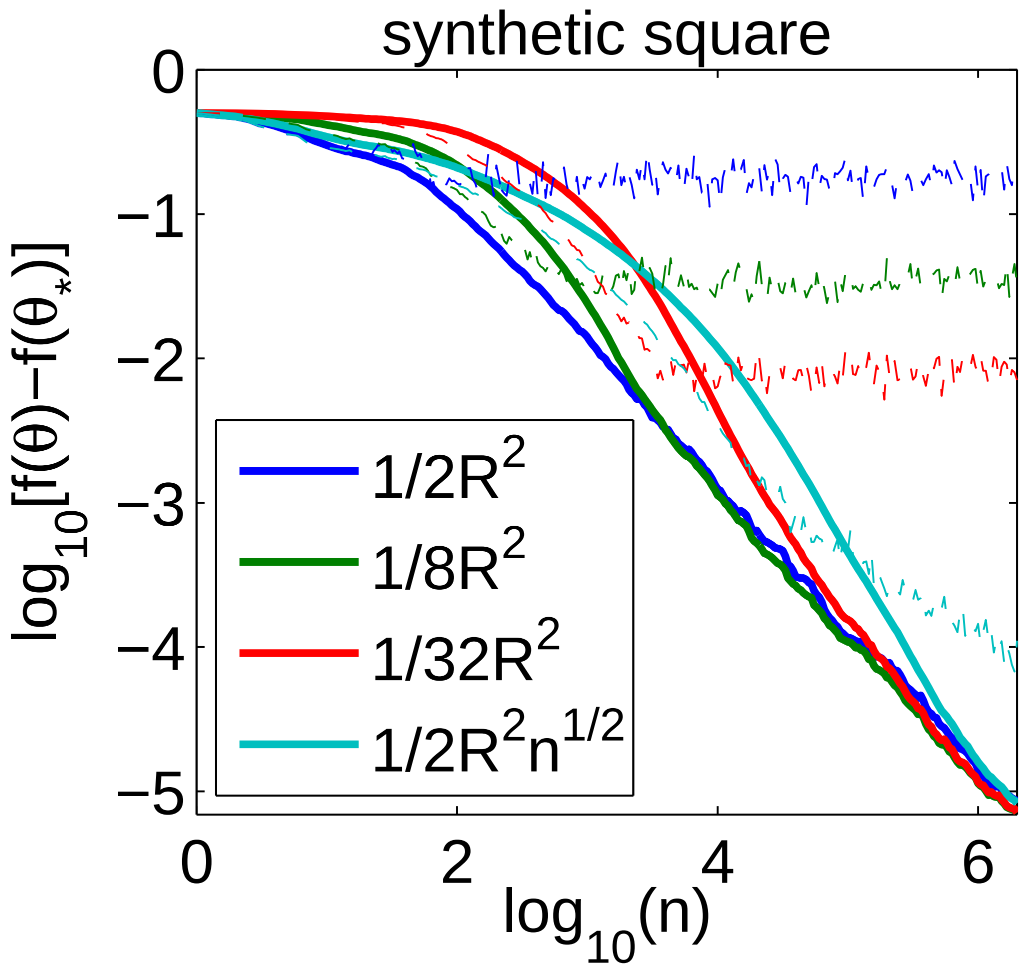


- **Explaining actual behavior for all n**

Bias-variance decomposition (Défossez and Bach, 2015)

- Simplification: dominating (but exact) term when $n \to \infty$ and $\gamma \to 0$
- **Variance** (e.g., starting from the solution)

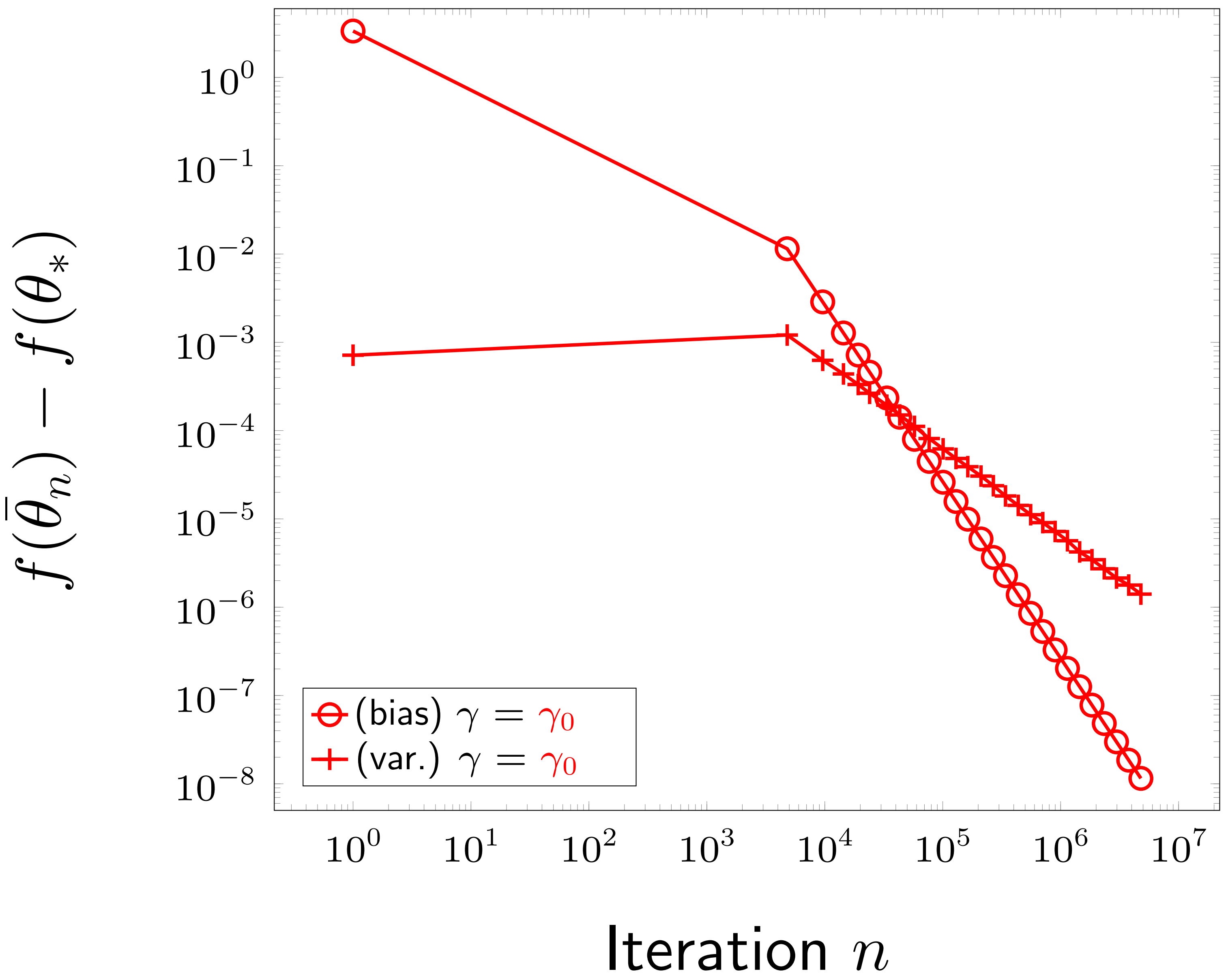
$$f(\bar{\theta}_n) - f(\theta_*) \sim \frac{1}{n} \mathbb{E}\Big[\varepsilon^2 \, \Phi(x)^\top H^{-1} \Phi(x)\Big]$$

  – NB: if noise ε is independent, then we obtain $\frac{d\sigma^2}{n}$
  – Exponentially decaying remainder terms (strongly convex problems)

- **Bias** (e.g., no noise)

$$f(\bar{\theta}_n) - f(\theta_*) \sim \frac{1}{n^2\gamma^2} (\theta_0 - \theta_*)^\top H^{-1} (\theta_0 - \theta_*)$$

Bias-variance decomposition (synthetic data $d = 25$)



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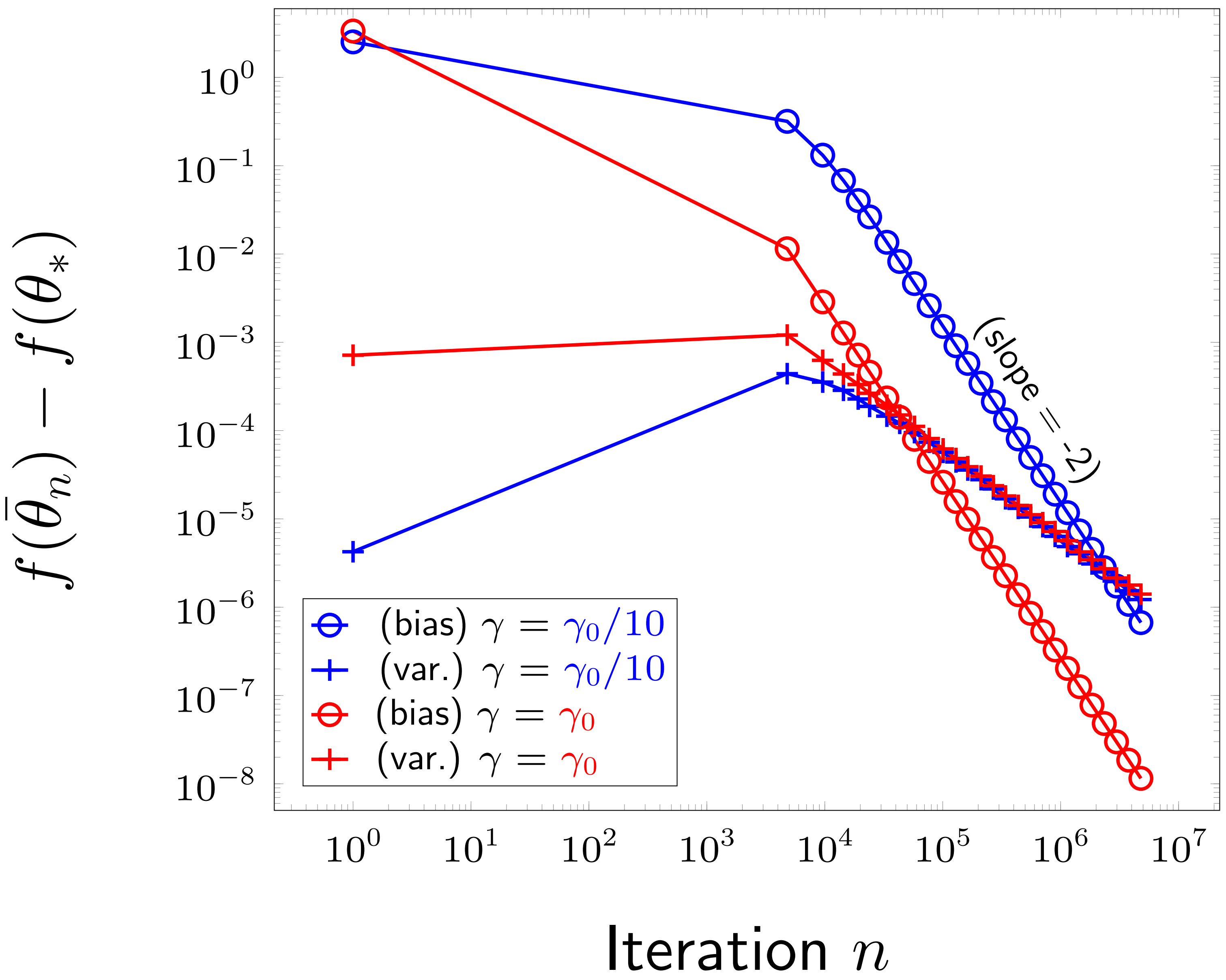

Optimal sampling (Défossez and Bach, 2015)

- **Sampling from a different distribution with importance weights**

$$\mathbb{E}_{p(x)p(y|x)}|y - \Phi(x)^\top \theta|^2 = \mathbb{E}_{q(x)p(y|x)} \frac{dp(x)}{dq(x)} |y - \Phi(x)^\top \theta|^2$$

  – Recursion: $\theta_n = \theta_{n-1} - \gamma \frac{dp(x_n)}{dq(x_n)} \big(\Phi(x_n)^\top \theta_{n-1} - y_n\big) \Phi(x_n)$

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  - Specific to least-squares $= \mathbb{E}_{q(x)p(y|x)} \left| \sqrt{\frac{dp(x)}{dq(x)}} y - \sqrt{\frac{dp(x)}{dq(x)}} \Phi(x)^\top \theta \right|^2$
  - Reweighting of the data: same bounds apply!

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- **Optimal for variance**: $\displaystyle\frac{dq(x)}{dp(x)} \propto \sqrt{\Phi(x)^\top H^{-1}\Phi(x)}$
  - Same density as active learning (Kanamori and Shimodaira, 2003)
  - Limited gains: different between first and second moments
  - Caveat: need to know H

Optimal sampling (Défossez and Bach, 2015)

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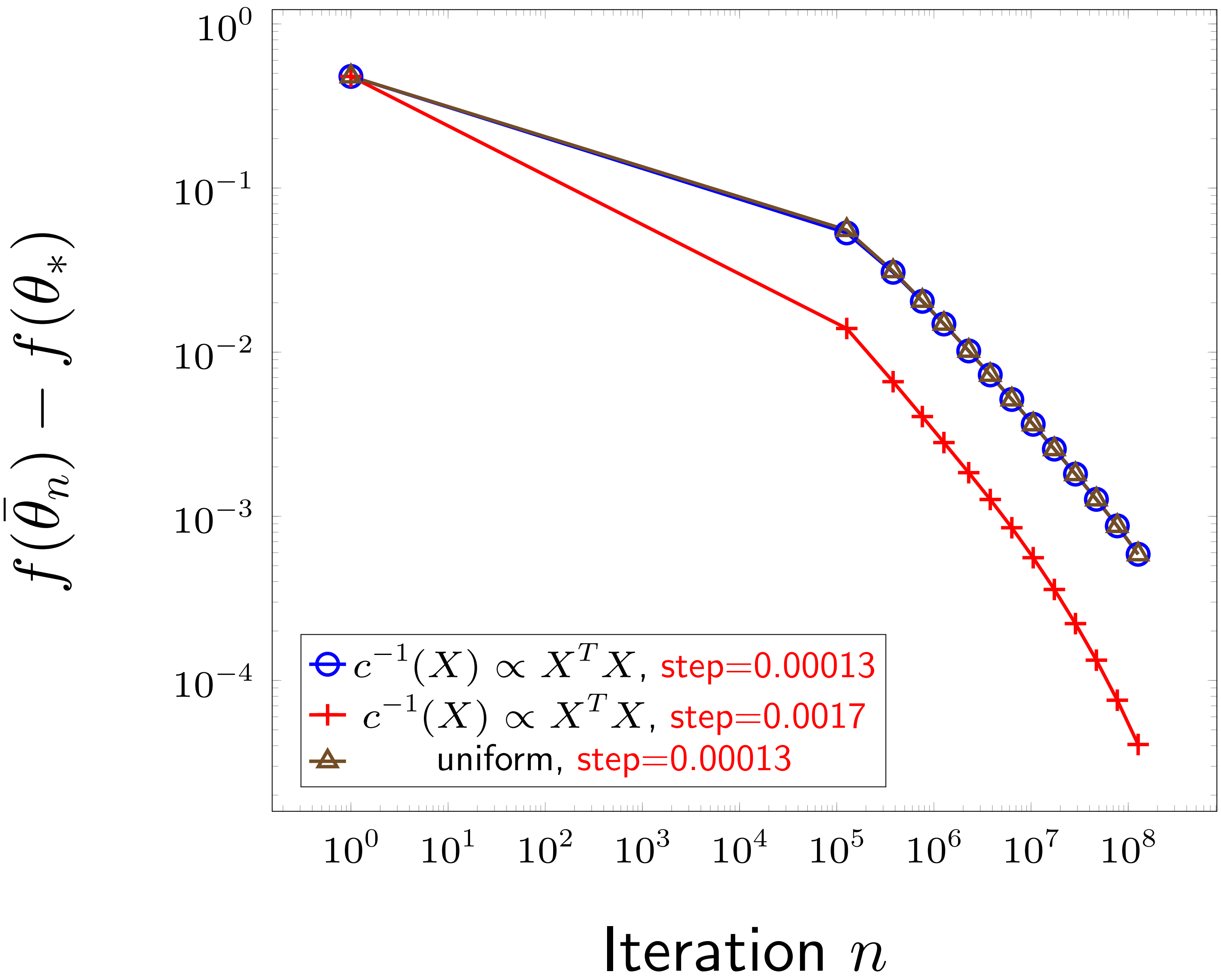
- **Optimal for bias**: $\dfrac{dq(x)}{dp(x)} \propto \|\Phi(x)\|^2$
  - Simpy allows biggest possible step size $\gamma < \frac{2}{\operatorname{tr} H}$
  - Large gains in practice
  - Corresponds to normalized least-mean-squares

Convergence on *Sido* dataset ($d = 4932$)



Achieving optimal bias and variance terms

- **Current results with averaged SGD**
  - **Variance** (starting from optimal θ_*) $= \dfrac{\sigma^2 d}{n}$
  - **Bias** (no noise) $= \min\left\{\dfrac{R^2\|\theta_0-\theta_*\|^2}{n}, \dfrac{R^4\langle\theta_0-\theta_*, H^{-1}(\theta_0-\theta_*)\rangle}{n^2}\right\}$

Achieving optimal bias and variance terms

- **Current results with averaged SGD** (ill-conditioned problems)
  - **Variance** (starting from optimal θ_*) $= \dfrac{\sigma^2 d}{n}$
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Achieving optimal bias and variance terms

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- **Acceleration is notoriously non-robust to noise** (d'Aspremont, 2008; Schmidt et al., 2011)
  - For non-structured noise, see Lan (2012)

Achieving optimal bias and variance terms

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Beyond least-squares - Markov chain interpretation

- Recursion $\theta_n = \theta_{n-1} - \gamma f_n'(\theta_{n-1})$ also defines a Markov chain
  - Stationary distribution π_γ such that $\int f'(\theta)\pi_\gamma(\mathrm{d}\theta) = 0$
  - When f' is not linear, $f'(\int \theta \pi_\gamma(\mathrm{d}\theta)) \neq \int f'(\theta)\pi_\gamma(\mathrm{d}\theta) = 0$

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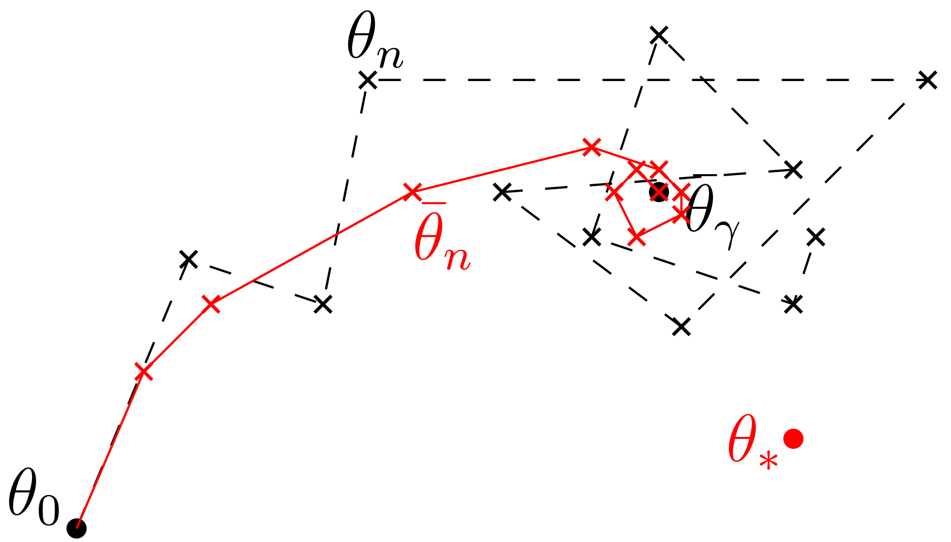
- **θ_n oscillates around the wrong value $\bar{\theta}_\gamma \neq \theta_*$**



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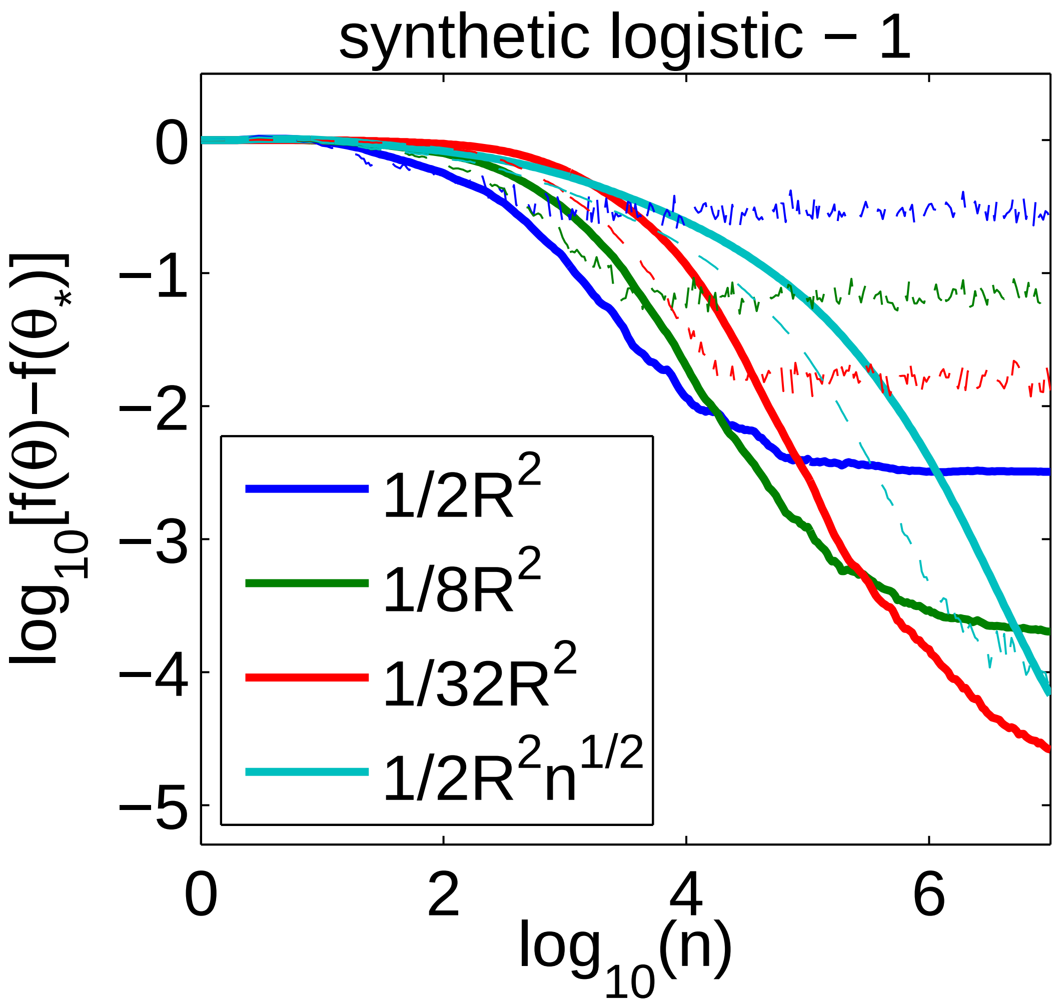
- θ_n **oscillates around the wrong value** $\bar{\theta}_\gamma \neq \theta_*$
  - moreover, $\|\theta_* - \theta_n\| = O_p(\sqrt{\gamma})$
  - Linear convergence up to the noise level for strongly-convex problems (Nedic and Bertsekas, 2000)

- **Ergodic theorem**
  - averaged iterates converge to $\bar{\theta}_\gamma \neq \theta_*$ at rate $O(1/n)$
  - moreover, $\|\theta_* - \bar{\theta}_\gamma\| = O(\gamma)$ (Bach, 2013)

Simulations - synthetic examples

- Gaussian distributions - $d = 20$



Restoring convergence through online Newton steps

- **Known facts**

  1. Averaged SGD with $\gamma_n \propto n^{-1/2}$ leads to *robust* rate $O(n^{-1/2})$ for all convex functions
  2. Averaged SGD with γ_n constant leads to *robust* rate $O(n^{-1})$ for all convex *quadratic* functions
  3. Newton's method squares the error at each iteration for smooth functions
  4. A single step of Newton's method is equivalent to minimizing the quadratic Taylor expansion

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  4. A single step of Newton's method is equivalent to minimizing the quadratic Taylor expansion

- **Online Newton step**

  – Rate: $O((n^{-1/2})^2 + n^{-1}) = O(n^{-1})$
  – Complexity: $O(d)$ per iteration

Restoring convergence through online Newton steps

- The Newton step for $f = \mathbb{E} f_n(\theta) \stackrel{\text{def}}{=} \mathbb{E}\big[\ell(y_n, \langle \theta, \Phi(x_n)\rangle)\big]$ at $\tilde{\theta}$ is equivalent to minimizing the quadratic approximation

$$
\begin{aligned}
g(\theta) &= f(\tilde{\theta}) + \langle f'(\tilde{\theta}), \theta - \tilde{\theta}\rangle + \tfrac{1}{2}\langle \theta - \tilde{\theta}, f''(\tilde{\theta})(\theta - \tilde{\theta})\rangle \\
&= f(\tilde{\theta}) + \langle \mathbb{E} f_n'(\tilde{\theta}), \theta - \tilde{\theta}\rangle + \tfrac{1}{2}\langle \theta - \tilde{\theta}, \mathbb{E} f_n''(\tilde{\theta})(\theta - \tilde{\theta})\rangle \\
&= \mathbb{E}\Big[f(\tilde{\theta}) + \langle f_n'(\tilde{\theta}), \theta - \tilde{\theta}\rangle + \tfrac{1}{2}\langle \theta - \tilde{\theta}, f_n''(\tilde{\theta})(\theta - \tilde{\theta})\rangle \Big]
\end{aligned}
$$

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- **Complexity of least-mean-square recursion for g is $O(d)$**

$$\theta_n = \theta_{n-1} - \gamma\big[f_n'(\tilde{\theta}) + f_n''(\tilde{\theta})(\theta_{n-1} - \tilde{\theta})\big]$$

  - $f_n''(\tilde{\theta}) = \ell''(y_n, \langle\tilde{\theta}, \Phi(x_n)\rangle)\Phi(x_n) \otimes \Phi(x_n)$ has rank one
  - New online Newton step without computing/inverting Hessians

Choice of support point for online Newton step

- **Two-stage procedure**

(1) Run $n/2$ iterations of averaged SGD to obtain $\tilde{\theta}$
(2) Run $n/2$ iterations of averaged constant step-size LMS

  - Reminiscent of one-step estimators (see, e.g., Van der Vaart, 2000)
  - Provable convergence rate of $O(d/n)$ for logistic regression
  - Additional assumptions but no strong convexity

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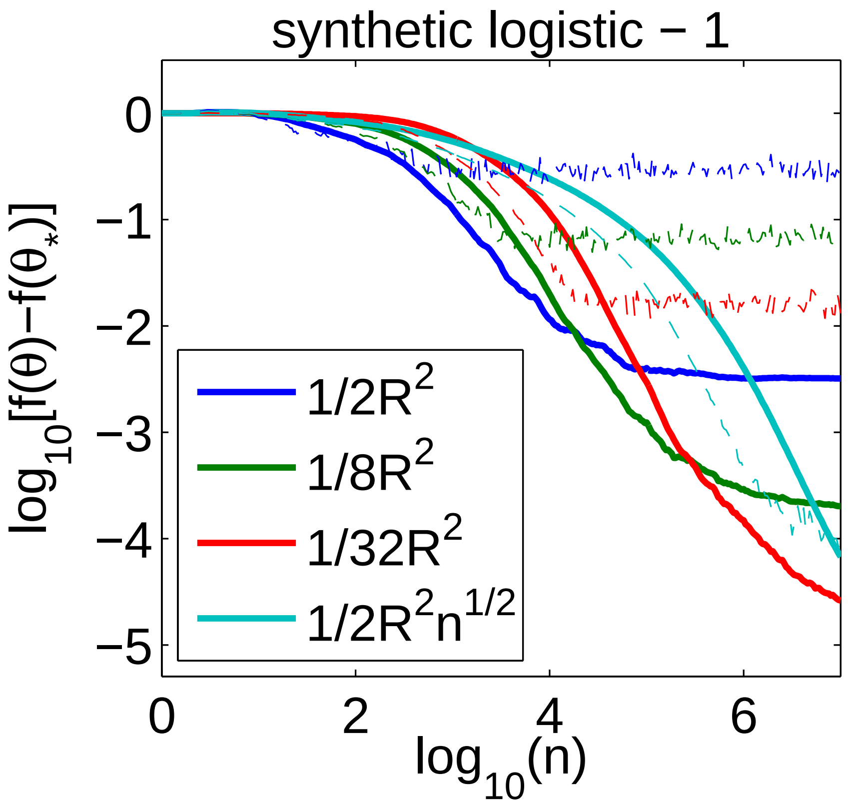
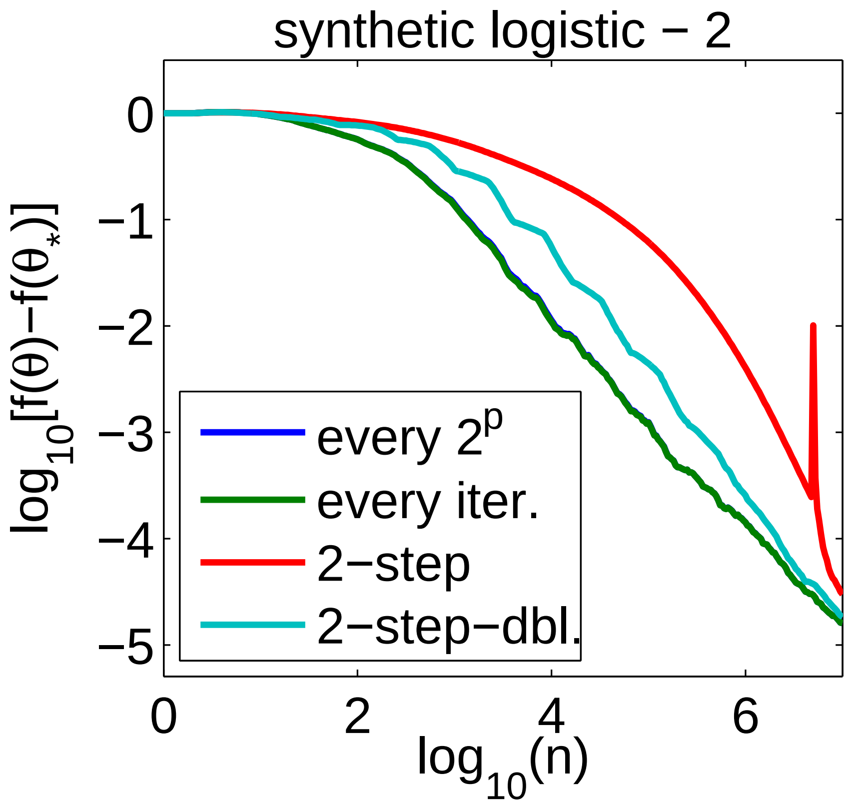
- **Update at each iteration using the current averaged iterate**
  - Recursion: $\boxed{\theta_n = \theta_{n-1} - \gamma\big[f_n'(\bar{\theta}_{n-1}) + f_n''(\bar{\theta}_{n-1})(\theta_{n-1} - \bar{\theta}_{n-1})\big]}$
  - No provable convergence rate (yet) but best practical behavior
  - Note (dis)similarity with regular SGD: $\theta_n = \theta_{n-1} - \gamma f_n'(\theta_{n-1})$

Simulations - synthetic examples

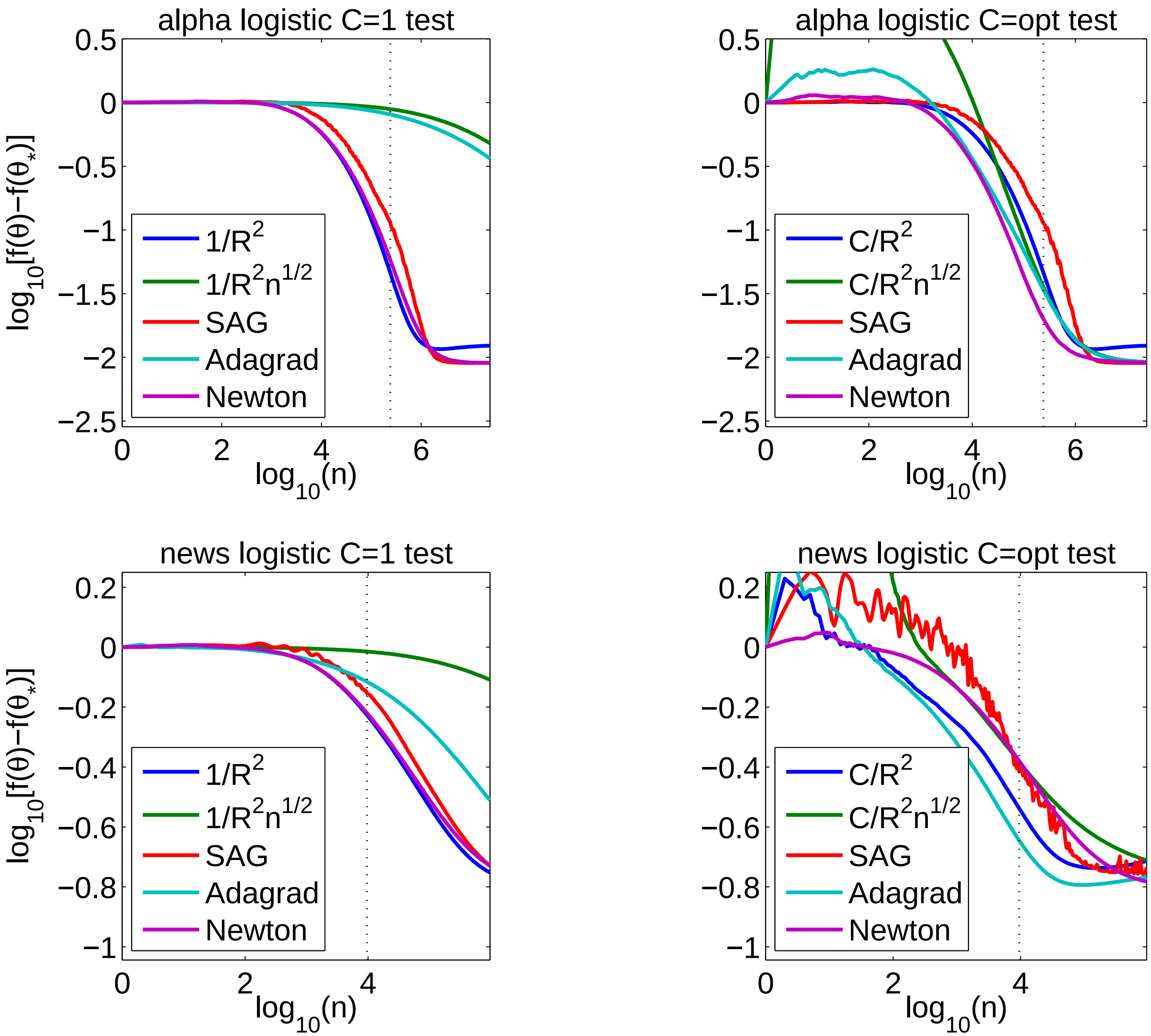
- Gaussian distributions - $d = 20$





Simulations - benchmarks

- *alpha* $(d = 500,\ n = 500\ 000)$, *news* $(d = 1\ 300\ 000,\ n = 20\ 000)$



Summary of rates of convergence

- Problem parameters
  - D diameter of the domain
  - B Lipschitz-constant
  - L smoothness constant
  - μ strong convexity constant

	convex	strongly convex
nonsmooth	deterministic: $BD/\sqrt{t}$ stochastic: $BD/\sqrt{n}$	deterministic: $B^2/(t\mu)$ stochastic: $B^2/(n\mu)$
smooth	deterministic: LD^2/t^2 stochastic: $LD^2/\sqrt{n}$	deterministic: $\exp(-t\sqrt{\mu/L})$ stochastic: $L/(n\mu)$
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Outline - I

1. **Introduction**
   - Large-scale machine learning and optimization
   - Classes of functions (convex, smooth, etc.)
   - Traditional statistical analysis through Rademacher complexity

2. **Classical methods for convex optimization**
   - Smooth optimization (gradient descent, Newton method)
   - Non-smooth optimization (subgradient descent)

3. **Classical stochastic approximation** (not covered)
   - Robbins-Monro algorithm (1951)

Outline - II

4. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds

5. **Smooth stochastic approximation algorithms**
   - Non-asymptotic analysis for smooth functions
   - Least-squares regression without decaying step-sizes

6. **Finite data sets**
   - Gradient methods with exponential convergence rates

Going beyond a single pass over the data

- **Stochastic approximation**
  - Assumes infinite data stream
  - Observations are used only once
  - Directly minimizes testing cost $\mathbb{E}_{(x,y)}\, \ell(y, \theta^\top \Phi(x))$

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- **Machine learning practice**
  - Finite data set $(x_1, y_1, \dots, x_n, y_n)$
  - Multiple passes
  - Minimizes training cost $\frac{1}{n} \sum_{i=1}^n \ell(y_i, \theta^\top \Phi(x_i))$
  - Need to regularize (e.g., by the ℓ_2-norm) to avoid overfitting

- **Goal**: minimize $g(\theta) = \dfrac{1}{n} \displaystyle\sum_{i=1}^n f_i(\theta)$

Stochastic vs. deterministic methods

- Minimizing $g(\theta) = \frac{1}{n}\sum_{i=1}^{n} f_i(\theta)$ with $f_i(\theta) = \ell\big(y_i, \theta^\top \Phi(x_i)\big) + \mu\Omega(\theta)$

- Batch gradient descent: $\theta_t = \theta_{t-1} - \gamma_t g'(\theta_{t-1}) = \theta_{t-1} - \frac{\gamma_t}{n}\sum_{i=1}^{n} f_i'(\theta_{t-1})$
  - Linear (e.g., exponential) convergence rate in $O(e^{-\alpha t})$
  - Iteration complexity is linear in n *(with line search)*

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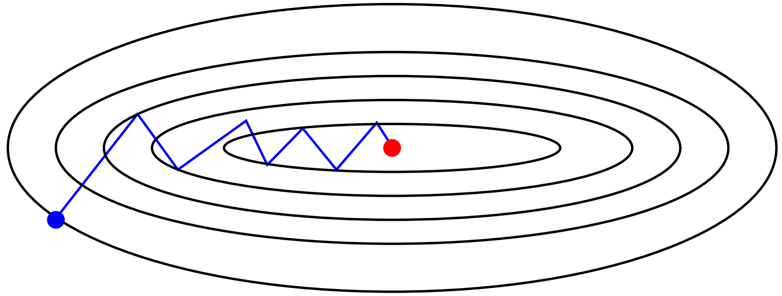

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- Stochastic gradient descent: $\theta_t = \theta_{t-1} - \gamma_t f'_{i(t)}(\theta_{t-1})$
  - Sampling with replacement: $i(t)$ random element of $\{1, \dots, n\}$
  - Convergence rate in $O(1/t)$
  - Iteration complexity is independent of n *(step size selection?)*

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- Minimizing $g(\theta) = \frac{1}{n}\sum_{i=1}^{n} f_i(\theta)$ with $f_i(\theta) = \ell\big(y_i, \theta^\top \Phi(x_i)\big) + \mu\Omega(\theta)$

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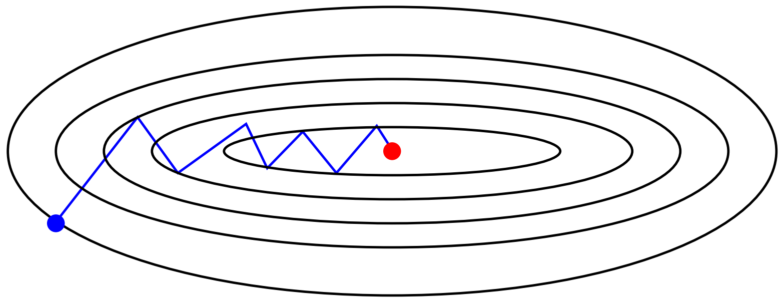

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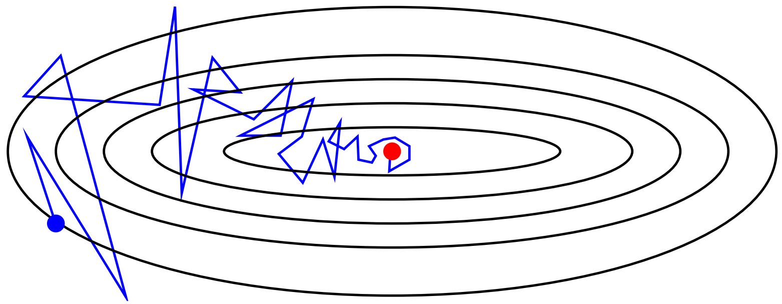
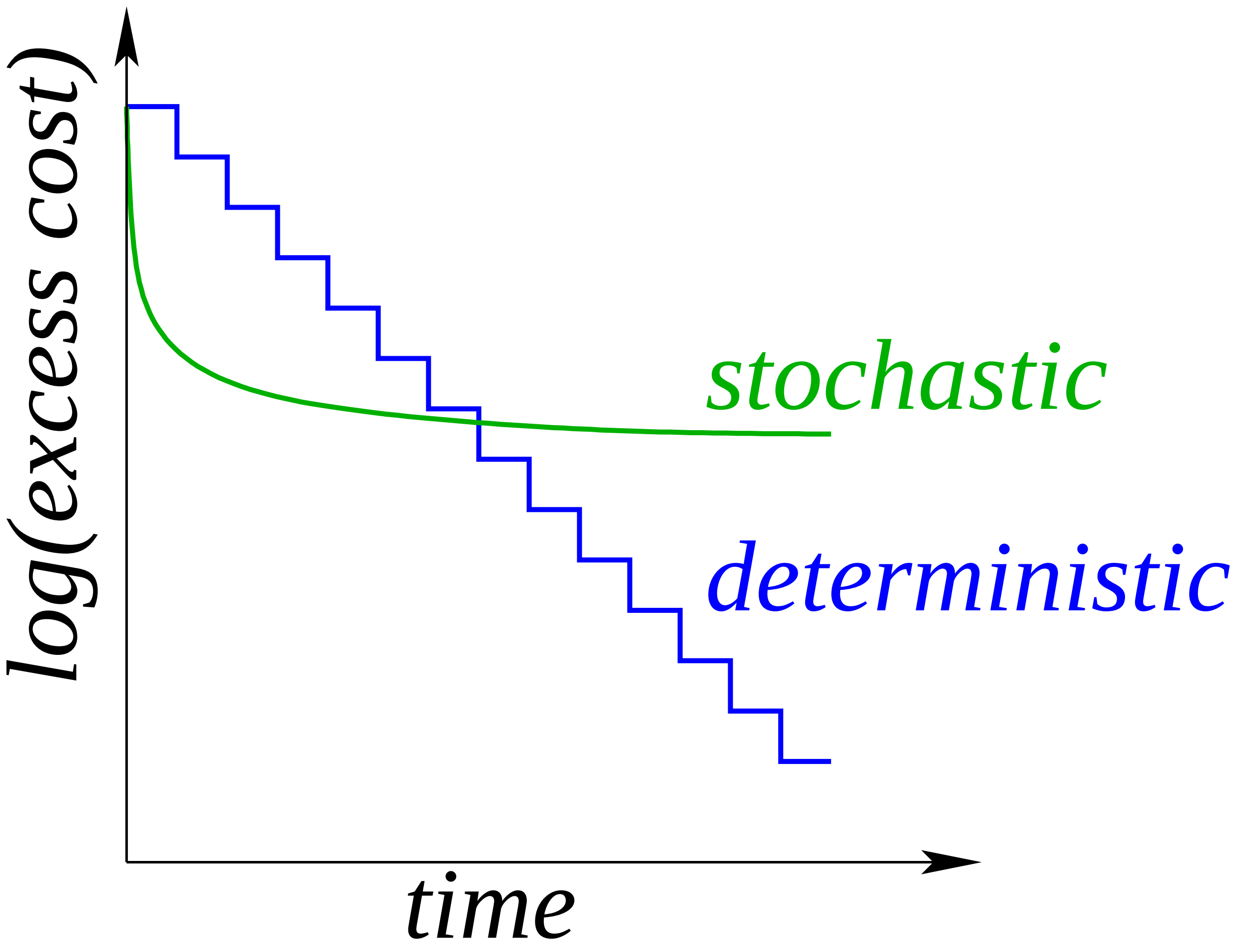


Stochastic vs. deterministic methods

- **Goal** = **best of both worlds**: Linear rate with $O(1)$ iteration cost
  Robustness to step size



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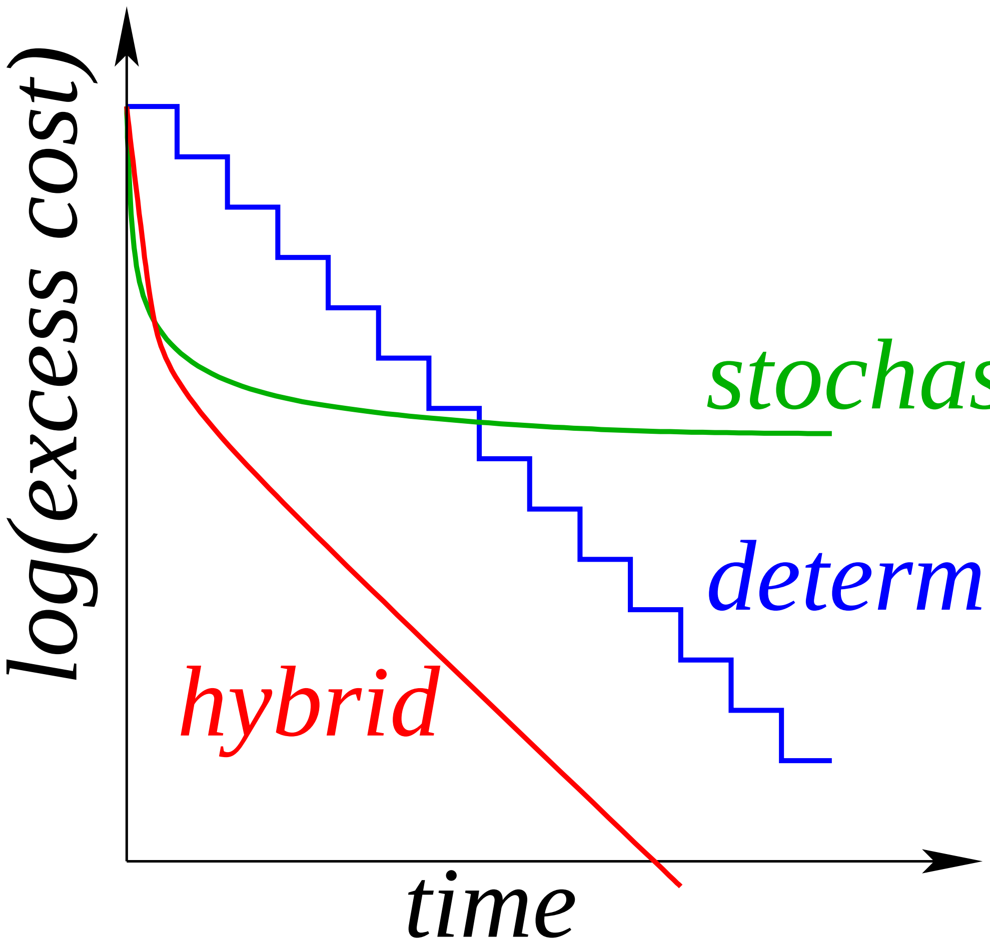

Stochastic average gradient (Le Roux, Schmidt, and Bach, 2012)

- **Stochastic average gradient** (SAG) iteration
  - Keep in memory the gradients of all functions f_i, $i = 1, \dots, n$
  - Random selection $i(t) \in \{1, \dots, n\}$ with replacement
  - Iteration: $\theta_t = \theta_{t-1} - \frac{\gamma_t}{n} \sum_{i=1}^{n} y_i^t$ with $y_i^t = \begin{cases} f_i'(\theta_{t-1}) & \text{if } i = i(t) \\ y_i^{t-1} & \text{otherwise} \end{cases}$

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- Stochastic version of incremental average gradient (Blatt et al., 2008)

- Extra memory requirement
  - Supervised machine learning
    - If $f_i(\theta) = \ell_i(y_i, \Phi(x_i)^\top \theta)$, then $f_i'(\theta) = \ell_i'(y_i, \Phi(x_i)^\top \theta)\, \Phi(x_i)$
    - Only need to store n real numbers

Stochastic average gradient - Convergence analysis

- **Assumptions**
  - Each f_i is R^2-smooth, $i = 1, \dots, n$
  - $g = \frac{1}{n}\sum_{i=1}^{n} f_i$ is μ-strongly convex (with potentially $\mu = 0$)
  - constant step size $\gamma_t = 1/(16R^2)$
  - initialization with one pass of averaged SGD

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- **Strongly convex case** (Le Roux et al., 2012, 2013)

$$\mathbb{E}\big[g(\theta_t) - g(\theta_*)\big] \leqslant \Big(\frac{8\sigma^2}{n\mu} + \frac{4R^2\|\theta_0 - \theta_*\|^2}{n}\Big) \exp\Big(-t \min\Big\{\frac{1}{8n}, \frac{\mu}{16R^2}\Big\}\Big)$$

  - Linear (exponential) convergence rate with $O(1)$ iteration cost
  - After one pass, reduction of cost by $\exp\Big(-\min\Big\{\frac{1}{8}, \frac{n\mu}{16R^2}\Big\}\Big)$

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- **Non-strongly convex case** (Le Roux et al., 2013)

$$\mathbb{E}\big[g(\theta_t) - g(\theta_*)\big] \leqslant 48 \frac{\sigma^2 + R^2\|\theta_0 - \theta_*\|^2}{\sqrt{n}} \frac{n}{t}$$

  - Improvement over regular batch and stochastic gradient
  - Adaptivity to potentially hidden strong convexity

Convergence analysis - Proof sketch

- **Main step**: find "good" Lyapunov function $J(\theta_t, y_1^t, \dots, y_n^t)$
  - such that $\mathbb{E}\big[J(\theta_t, y_1^t, \dots, y_n^t)|\mathcal{F}_{t-1}\big] < J(\theta_{t-1}, y_1^{t-1}, \dots, y_n^{t-1})$
  - no natural candidates

- **Computer-aided proof**
  - Parameterize function $J(\theta_t, y_1^t, \dots, y_n^t) = g(\theta_t) - g(\theta_*) + \text{quadratic}$
  - Solve semidefinite program to obtain candidates (that depend on n, μ, L)
  - Check validity with symbolic computations

Rate of convergence comparison

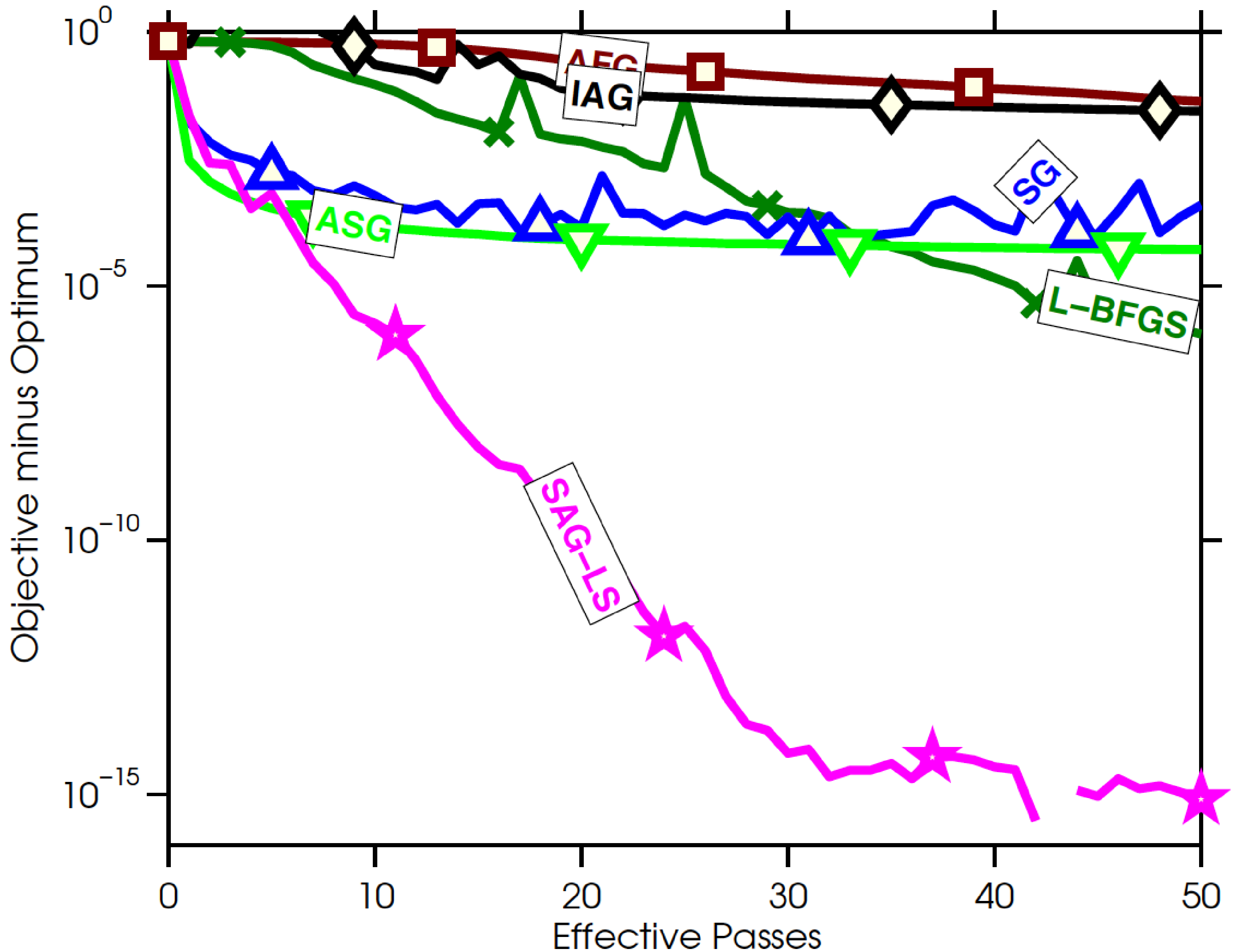
- Assume that $L = 100$, $\mu = .01$, and $n = 80000$ ($L \neq R^2$)
  - Full gradient method has rate
  $$\left(1 - \frac{\mu}{L}\right) = 0.9999$$
  - Accelerated gradient method has rate
  $$\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.9900$$
  - Running n iterations of SAG for the same cost has rate
  $$\left(1 - \frac{1}{8n}\right)^n = 0.8825$$
  - *Fastest possible* first-order method has rate
  $$\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.9608$$

- **Beating two lower bounds** (with additional assumptions)
  - (1) stochastic gradient and (2) full gradient

Stochastic average gradient
Implementation details and extensions

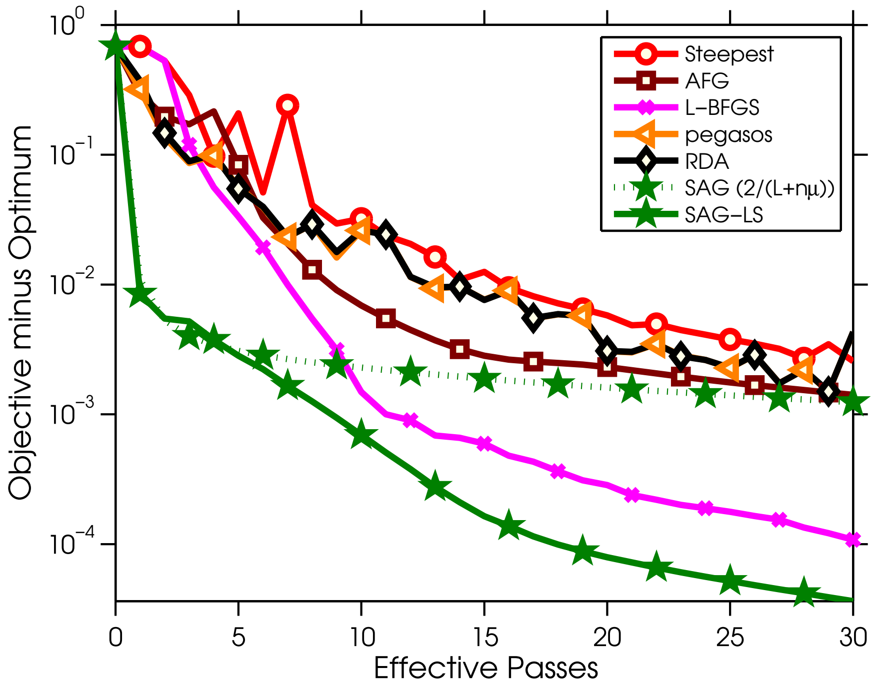
- The algorithm can use sparsity in the features to reduce the storage and iteration cost
- Grouping functions together can further reduce the memory requirement
- We have obtained good performance when R^2 is not known with a heuristic line-search
- Algorithm allows non-uniform sampling
- Possibility of making proximal, coordinate-wise, and Newton-like variants

`spam` dataset (n = 92 189, d = 823 470)

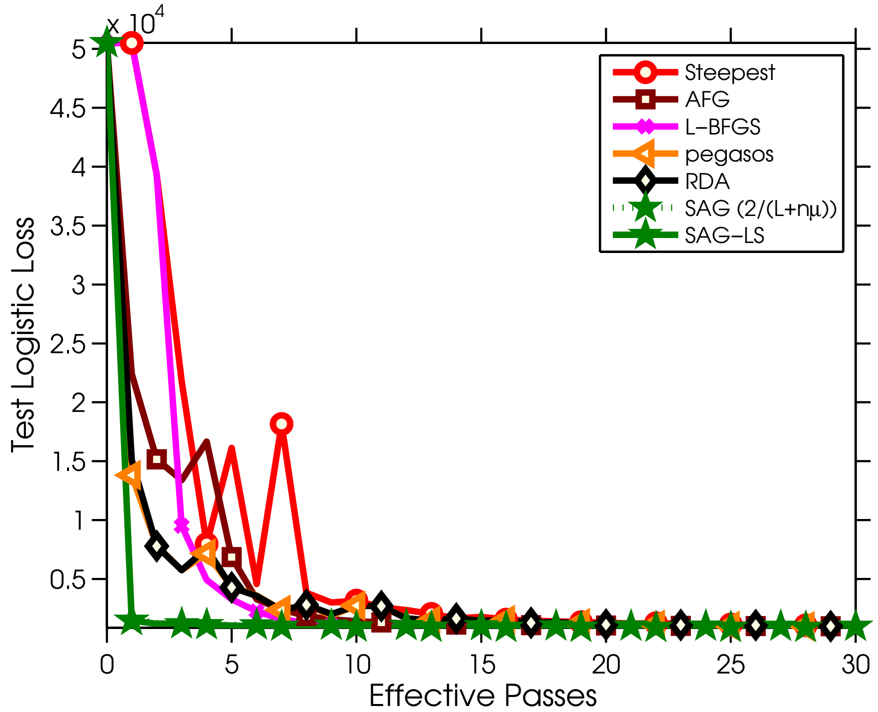


protein dataset ($n = 145751$, $d = 74$)

- Dataset split in two (training/testing)



Training cost



Testing cost

Extensions and related work

- **Exponential convergence rate for strongly convex problems**
- **Need to store gradients**
  - SVRG (Johnson and Zhang, 2013)
- **Adaptivity to non-strong convexity**
  - SAGA (Defazio, Bach, and Lacoste-Julien, 2014)
- **Simple proof**
  - SVRG, SAGA, random coordinate descent (Nesterov, 2012; Shalev-Shwartz and Zhang, 2012)
- **Lower bounds**
  - Agarwal and Bottou (2014)

Variance reduction

- **Principle**: reducing variance of sample of X by using a sample from another random variable Y with known expectation

$$Z_\alpha = \alpha(X - Y) + \mathbb{E}Y$$

  - $\mathbb{E}Z_\alpha = \alpha\mathbb{E}X + (1-\alpha)\mathbb{E}Y$
  - $\operatorname{var} Z_\alpha = \alpha^2\big[\operatorname{var} X + \operatorname{var} Y - 2\operatorname{cov}(X,Y)\big]$
  - $\alpha = 1$: no bias, $\alpha < 1$: potential bias (but reduced variance)
  - Useful if Y positively correlated with X

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- **Principle**: reducing variance of sample of X by using a sample from another random variable Y with known expectation

$$Z_\alpha = \alpha(X - Y) + \mathbb{E}Y$$

  – $\mathbb{E}Z_\alpha = \alpha\mathbb{E}X + (1-\alpha)\mathbb{E}Y$
  – $\operatorname{var} Z_\alpha = \alpha^2\big[\operatorname{var} X + \operatorname{var} Y - 2\operatorname{cov}(X,Y)\big]$
  – $\alpha = 1$: no bias, $\alpha < 1$: potential bias (but reduced variance)
  – Useful if Y positively correlated with X

- **Application to gradient estimation : SVRG** (Johnson and Zhang, 2013)

  – Estimating the averaged gradient $g'(\theta) = \frac{1}{n}\sum_{i=1}^n f_i'(\theta)$
  – Using the gradients of a previous iterate $\tilde{\theta}$

Stochastic variance reduced gradient (SVRG)

- **Algorithm divide into "epochs"**

- At each epoch, starting from $\theta_0 = \tilde{\theta}$, perform the iteration
  - Sample i_t uniformly at random
  - Gradient step: $\theta_t = \theta_{t-1} - \gamma \Big[f'_{i_t}(\theta_{t-1}) - f'_{i_t}(\tilde{\theta}) + g'(\tilde{\theta}) \Big]$

- **Proposition**: If each f_i is R^2-smooth and $g = \frac{1}{n}\sum_{i=1}^n f_i$ is μ-strongly convex, then after $k = 20R^2/\mu$ steps and with $\gamma = 1/10R^2$, then $f(\theta) - f(\theta_*)$ is reduced by 10%

Outline - I

1. **Introduction**
   - Large-scale machine learning and optimization
   - Classes of functions (convex, smooth, etc.)
   - Traditional statistical analysis through Rademacher complexity

2. **Classical methods for convex optimization**
   - Smooth optimization (gradient descent, Newton method)
   - Non-smooth optimization (subgradient descent)

3. **Classical stochastic approximation** (not covered)
   - Robbins-Monro algorithm (1951)

Outline - II

4. **Non-smooth stochastic approximation**
   - Stochastic (sub)gradient and averaging
   - Non-asymptotic results and lower bounds

5. **Smooth stochastic approximation algorithms**
   - Non-asymptotic analysis for smooth functions
   - Least-squares regression without decaying step-sizes

6. **Finite data sets**
   - Gradient methods with exponential convergence rates

Subgradient descent for machine learning

- **Assumptions** (f is the expected risk, $\hat{f}$ the empirical risk)
  - "Linear" predictors: $\theta(x) = \theta^\top \Phi(x)$, with $\|\Phi(x)\|_2 \leqslant R$ a.s.
  - $\hat{f}(\theta) = \frac{1}{n}\sum_{i=1}^n \ell(y_i, \Phi(x_i)^\top \theta)$
  - G-Lipschitz loss: f and $\hat{f}$ are GR-Lipschitz on $\Theta = \{\|\theta\|_2 \leqslant D\}$
- **Statistics**: with probability greater than $1-\delta$

$$\sup_{\theta \in \Theta} |\hat{f}(\theta) - f(\theta)| \leqslant \frac{GRD}{\sqrt{n}}\left[2 + \sqrt{2 \log \frac{2}{\delta}}\,\right]$$

- **Optimization**: after t iterations of subgradient method

$$\hat{f}(\hat{\theta}) - \min_{\eta \in \Theta} \hat{f}(\eta) \leqslant \frac{GRD}{\sqrt{t}}$$

- $t = n$ iterations, with total running-time complexity of $O(n^2 d)$

Stochastic subgradient "descent"/method

- **Assumptions**
  - f_n convex and B-Lipschitz-continuous on $\{\|\theta\|_2 \leqslant D\}$
  - (f_n) i.i.d. functions such that $\mathbb{E} f_n = f$
  - θ_* global optimum of f on $\{\|\theta\|_2 \leqslant D\}$

- **Algorithm**: $\theta_n = \Pi_D\left(\theta_{n-1} - \frac{2D}{B\sqrt{n}} f_n'(\theta_{n-1})\right)$

- **Bound**:

$$\mathbb{E} f\left(\frac{1}{n}\sum_{k=0}^{n-1}\theta_k\right) - f(\theta_*) \leqslant \frac{2DB}{\sqrt{n}}$$

- "Same" three-line proof as in the deterministic case

- Minimax rate (Nemirovsky and Yudin, 1983; Agarwal et al., 2012)

- Running-time complexity: $O(dn)$ after n iterations

Summary of new results (Bach and Moulines, 2011)

- Stochastic gradient descent with learning rate $\gamma_n = Cn^{-\alpha}$

- **Strongly convex smooth objective functions**
  - Old: $O(n^{-1})$ rate achieved without averaging for $\alpha = 1$
  - New: $O(n^{-1})$ rate achieved with averaging for $\alpha \in [1/2, 1]$
  - Non-asymptotic analysis with explicit constants
  - Forgetting of initial conditions
  - Robustness to the choice of C

- **Convergence rates** for $\mathbb{E}\|\theta_n - \theta_*\|^2$ and $\mathbb{E}\|\bar{\theta}_n - \theta_*\|^2$
  - no averaging: $O\Big(\frac{\sigma^2\gamma_n}{\mu}\Big) + O(e^{-\mu n\gamma_n})\|\theta_0 - \theta_*\|^2$
  - averaging: $\frac{\operatorname{tr} H(\theta_*)^{-1}}{n} + \mu^{-1}O(n^{-2\alpha} + n^{-2+\alpha}) + O\Big(\frac{\|\theta_0 - \theta_*\|^2}{\mu^2 n^2}\Big)$

Least-mean-square algorithm

- **Least-squares**: $f(\theta) = \frac{1}{2}\mathbb{E}\big[(y_n - \langle \Phi(x_n), \theta\rangle)^2\big]$ with $\theta \in \mathbb{R}^d$
  - SGD = least-mean-square algorithm (see, e.g., Macchi, 1995)
  - usually studied without averaging and decreasing step-sizes
  - with strong convexity assumption $\mathbb{E}\big[\Phi(x_n) \otimes \Phi(x_n)\big] = H \succcurlyeq \mu \cdot \mathrm{Id}$

- **New analysis for averaging and constant step-size** $\gamma = 1/(4R^2)$
  - Assume $\|\Phi(x_n)\| \leqslant R$ and $|y_n - \langle \Phi(x_n), \theta_*\rangle| \leqslant \sigma$ almost surely
  - No assumption regarding lowest eigenvalues of H
  - Main result: $\boxed{\mathbb{E} f(\bar{\theta}_{n-1}) - f(\theta_*) \leqslant \frac{4\sigma^2 d}{n} + \frac{4R^2\|\theta_0 - \theta_*\|^2}{n}}$

- **Matches statistical lower bound** (Tsybakov, 2003)
  - Non-asymptotic robust version of Györfi and Walk (1996)

Choice of support point for online Newton step

- **Two-stage procedure**

(1) Run $n/2$ iterations of averaged SGD to obtain $\tilde{\theta}$
(2) Run $n/2$ iterations of averaged constant step-size LMS

  - Reminiscent of one-step estimators (see, e.g., Van der Vaart, 2000)
  - Provable convergence rate of $O(d/n)$ for logistic regression
  - Additional assumptions but no strong convexity

- **Update at each iteration using the current averaged iterate**

  - Recursion: $$\boxed{\theta_n = \theta_{n-1} - \gamma\big[f_n'(\bar{\theta}_{n-1}) + f_n''(\bar{\theta}_{n-1})(\theta_{n-1} - \bar{\theta}_{n-1})\big]}$$
  - No provable convergence rate (yet) but best practical behavior
  - Note (dis)similarity with regular SGD: $\theta_n = \theta_{n-1} - \gamma f_n'(\theta_{n-1})$

Stochastic average gradient (Le Roux, Schmidt, and Bach, 2012)

- **Stochastic average gradient** (SAG) iteration
  - Keep in memory the gradients of all functions f_i, $i = 1, \dots, n$
  - Random selection $i(t) \in \{1, \dots, n\}$ with replacement
  - Iteration: $\theta_t = \theta_{t-1} - \frac{\gamma_t}{n} \sum_{i=1}^{n} y_i^t$ with $y_i^t = \begin{cases} f_i'(\theta_{t-1}) & \text{if } i = i(t) \\ y_i^{t-1} & \text{otherwise} \end{cases}$

- Stochastic version of incremental average gradient (Blatt et al., 2008)

- Extra memory requirement
  - Supervised machine learning
    - If $f_i(\theta) = \ell_i(y_i, \Phi(x_i)^\top \theta)$, then $f_i'(\theta) = \ell_i'(y_i, \Phi(x_i)^\top \theta)\, \Phi(x_i)$
    - Only need to store n real numbers

Summary of rates of convergence

- Problem parameters
  - D diameter of the domain
  - B Lipschitz-constant
  - L smoothness constant
  - μ strong convexity constant

	convex	strongly convex
nonsmooth	deterministic: $BD/\sqrt{t}$ stochastic: $BD/\sqrt{n}$	deterministic: $B^2/(t\mu)$ stochastic: $B^2/(n\mu)$
smooth	deterministic: LD^2/t^2 stochastic: $LD^2/\sqrt{n}$ finite sum: n/t	deterministic: $\exp(-t\sqrt{\mu/L})$ stochastic: $L/(n\mu)$ finite sum: $\exp(-\min\{1/n, \mu/L\}t)$
quadratic	deterministic: LD^2/t^2 stochastic: $d/n + LD^2/n$	deterministic: $\exp(-t\sqrt{\mu/L})$ stochastic: $d/n + LD^2/n$

Conclusions
Machine learning and convex optimization

- **Statistics with or without optimization?**
  - Significance of mixing algorithms with analysis
  - Benefits of mixing algorithms with analysis

- **Open problems**
  - Non-parametric stochastic approximation
  - Characterization of implicit regularization of online methods
  - Structured prediction
  - Going beyond a single pass over the data (testing performance)
  - Further links between convex optimization and online learning/bandits
  - Parallel and distributed optimization

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