# MVA "Kernel methods"
# Homework 5

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**Exercice 1. MKL on a DAG**

Let $V = (v_1, \ldots, v_M)$ be the vertices of a directed acyclic graph (DAG). For any $v \in V$, we denote by $D(v) \subset V$ the set of descendants of $v$ (including itself), and let $d_v \geq 0$ be a weight associated to each vertex $v$. We assume that to each vertex $v \in V$ is associated a positive definite kernel $K_v$ over a space $\mathcal{X}$.

**a.** Using the notations of the course (slide 159), show that the following *weighted* MKL with the set of kernels $\{K_v : v \in V\}$:

$$\min_{(f_{v_1},\ldots,f_{v_M}) \in \mathcal{H}_{K_{v_1}} \times \ldots \times \mathcal{H}_{K_{v_M}}} \left\{ R\left(\sum_{v \in V} f_v^n\right) + \lambda \left(\sum_{v \in V} d_v \| f_v \|_{\mathcal{H}_{K_v}}\right)^2 \right\}$$

is equivalent to solving:

$$\min_{\eta \in \Sigma} \min_{f \in \mathcal{H}_{K_\eta}} \left\{ R(f^n) + \lambda \| f \|^2_{\mathcal{H}_{K_\eta}} \right\}$$

for some set $\Sigma$ to be determined.

**b.** We now consider the following variant of MKL which takes the graph structure into account:

$$\min_{(f_{v_1},\ldots,f_{v_M}) \in \mathcal{H}_{K_{v_1}} \times \ldots \times \mathcal{H}_{K_{v_M}}} \left\{ R\left(\sum_{v \in V} f_v^n\right) + \lambda \left(\sum_{v \in V} d_v \left(\sum_{w \in D(v)} \| f_w \|^2_{\mathcal{H}_{K_w}}\right)^{\frac{1}{2}}\right)^2 \right\} . \tag{1}$$

Can you intuitively explain why we may want to do this, and what we can expect from the solution of this formulation?

**c.** Show that the MKL formulation (1) is equivalent to solving:

$$\min_{\eta \in \Sigma_V} \min_{f \in \mathcal{H}_{K_\eta}} \left\{ R(f^n) + \lambda \| f \|^2_{\mathcal{H}_{K_\eta}} \right\}$$

for some set $\Sigma_V$ to be determined.

**d.** Show that if the DAG is a tree, then $\Sigma_V$ is convex. Is it also convex for a general DAG?

**Exercice 2. Sobolev RKHS**

Show that the set

$$\mathcal{H} = \left\{ f : [0,1] \mapsto \mathbb{R}, \text{absolutely continuous}, f' \in L^2\left([0,1]\right), f\left(0\right) = f(1) = 0 \right\}$$

endowed with the bilinear form:

$$\forall \left(f,g\right) \in \mathcal{F}^2 \left\langle f,g \right\rangle_{\mathcal{H}} = \int_0^1 f'\left(u\right) g'\left(u\right) du$$

is an RKHS, and determine its reproducing kernel.