MVA "Kernel methods"
Homework 3

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Exercice 1.
Let $(x_1, y_1), \ldots, (x_n, y_n)$ a training set of examples where $x_i \in \mathcal{X}$, a space endowed with a positive definite kernel K, and $y_i \in \{-1, 1\}$, for $i = 1, \ldots, n$. $\mathcal{H}_K$ denotes the RKHS of the kernel K. We want to learn a function $f : \mathcal{X} \mapsto \mathbb{R}$ by solving the following optimization problem:

$$\min_{f \in \mathcal{H}_K} \frac{1}{n} \sum_{i=1}^{n} \ell_{y_i}\left(f(x_i)\right) \quad \text{such that} \quad \| f \|_{\mathcal{H}_K} \leq B \ , \tag{1}$$

where ℓ_y is a convex loss functions (for $y \in \{-1, 1\}$) and $B > 0$ is a parameter.
a. Show that there exists $\lambda \geq 0$ such that the solution to problem (1) can be found be solving the following problem:

$$\min_{\alpha \in \mathbb{R}^n} R(K\alpha) + \lambda \alpha^\top K \alpha \, , \tag{2}$$

where K is the $n \times n$ Gram matrix and $R : \mathbb{R}^n \mapsto \mathbb{R}$ should be explicited.
b. Compute the Fenchel-Legendre transform[1] R^* of R in terms of the Fenchel-Legendre transform ℓ_y^* of ℓ_y.

[1] For any function $f : \mathbb{R}^N \mapsto \mathbb{R}$, the *Fenchel-Legendre transform* (or *convex conjugate*) of f is the function $f^* : \mathbb{R}^N \mapsto \mathbb{R}$ defined by

$$f^*(u) = \sup_{x \in \mathbb{R}^N} < x, u > -f(x) \, .$$

c. Adding the slack variable $u = K\alpha$, the problem (1) can be rewritten as a constrained optimization problem:

$$\min_{\alpha \in \mathbb{R}^n, u \in \mathbb{R}^n} R(u) + \lambda \alpha^\top K \alpha \quad \text{such that} \quad u = K\alpha\,. \tag{3}$$

Express the dual problem of (3) in terms of R^*, and explain how a solution to (3) can be found from a solution to the dual problem.
c. Explicit the dual problem for the hinge loss:

$$\ell_y(u) = \max(0, 1 - yu)\,.$$

Exercice 2.
Let K_1 and K_2 be two positive definite kernels on a set $\mathcal{X}$. What is the RKHS of the kernel $K_1 + K_2$?