Homework 6

Jean-Philippe Vert

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1 B_n-splines

The convolution between two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f \star g(x) = \int_{-\infty}^{\infty} f(u) g(x-u) du,$$

when this integral exists.

Let now the function:

$$I(x) = \begin{cases} 1 & \text{si } -1 \leq x \leq 1, \\ 0 & \text{si } x < -1 \text{ ou } x > 1, \end{cases}$$

and $B_n = I^{\star n}$ for $n \in \mathbb{N}_*$ (that is, the function I convolved n times with itself: $B_1 = I, B_2 = I \star I, B_3 = I \star I \star I$, etc...).

Is the function $k(x,y) = B_n(x-y)$ a positive definite kernel over $\mathbb{R} \times \mathbb{R}$? If yes, describe the corresponding reproducing kernel Hilbert space.

2 More kernels...

3.1 Are the following functions positive definite kernels?

$$\forall x, y \in \mathbb{R}, \quad K_2(x,y) = \frac{1}{2 - e^{-\| x-y \|^2}}$$

$$\forall x, y \in \mathbb{R}, \quad K_3(x,y) = \max\left(0, 1 - |x-y|\right)$$

3.2. For any $n > 0$, show that the $n \times n$ Hankel matrix $A_{ij} = \frac{1}{1+i+j}$ is positive semidefinite.

3.3. Describe the functions $\phi : [0,1] \mapsto \mathbb{R}$ such that:

$$K(x,y) = \phi\left(\max(x+y-1,0)\right)$$

is a positive definite kernel on $[0,1]$.

3.4. (BONUS) Describe the functions $\phi : \mathbb{R}^+ \mapsto \mathbb{R}$ such that:

$$K(x,y) = \phi\left(\max(x,y)\right)$$

is a positive definite kernel on $\mathbb{R}^+$.