

Group lasso for genomic data

Jean-Philippe Vert

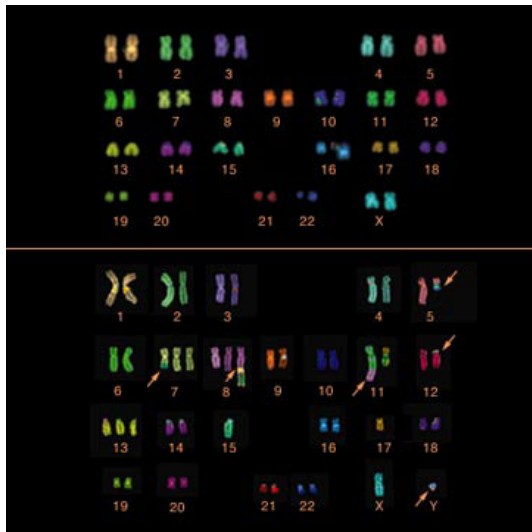
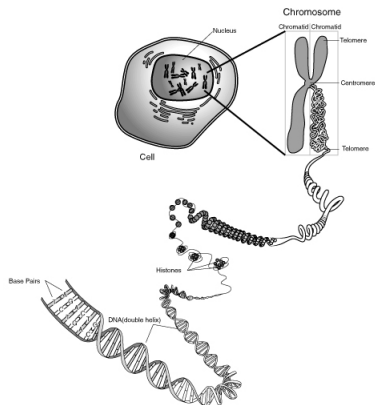
Mines ParisTech and Curie Institute

Machine learning: Theory and Computation workshop,
IMA, Minneapolis, March 26-30, 2012

- 1 Motivations
- 2 Finding multiple change-points in a single profile
- 3 Finding multiple change-points shared by many signals
- 4 Learning molecular classifiers with network information
- 5 Conclusion

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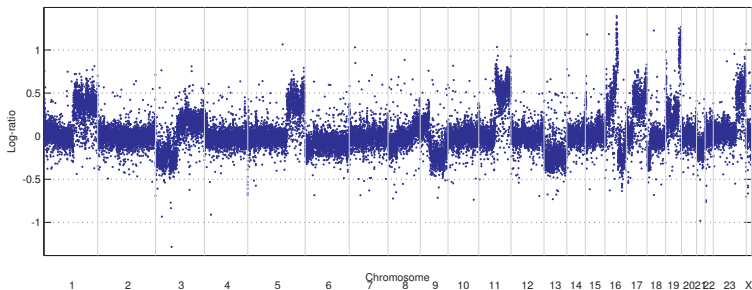
Chromosomal aberrations in cancer



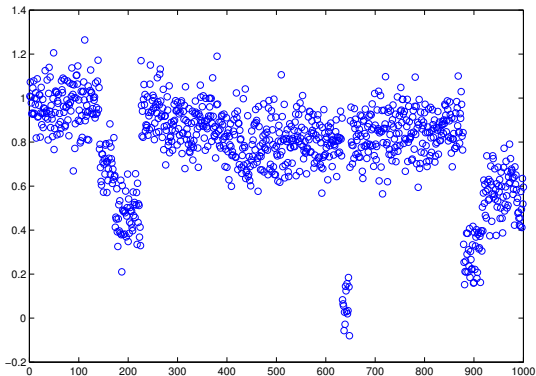
Comparative Genomic Hybridization (CGH)

Motivation

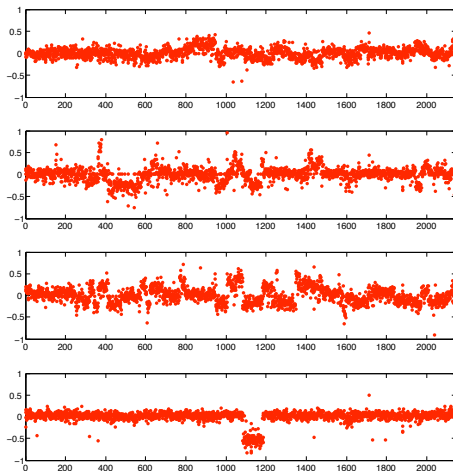
- Comparative genomic hybridization (CGH) data measure the **DNA copy number** along the genome
- Very useful, in particular in cancer research to observe systematically variants in DNA content



Can we identify breakpoints and "smooth" each profile?

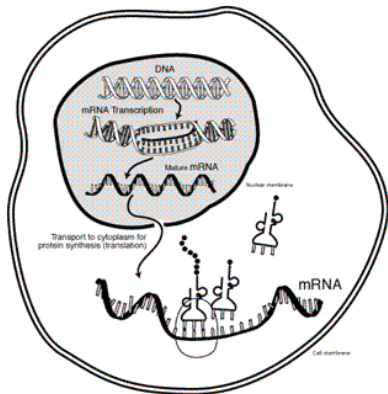


Can we detect frequent breakpoints?



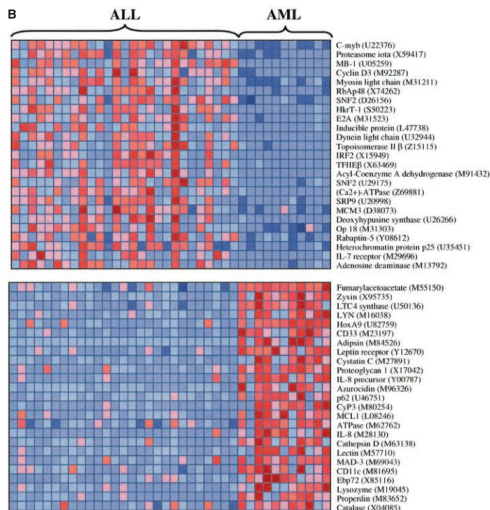
A collection of bladder tumour copy number profiles.

DNA → RNA → protein

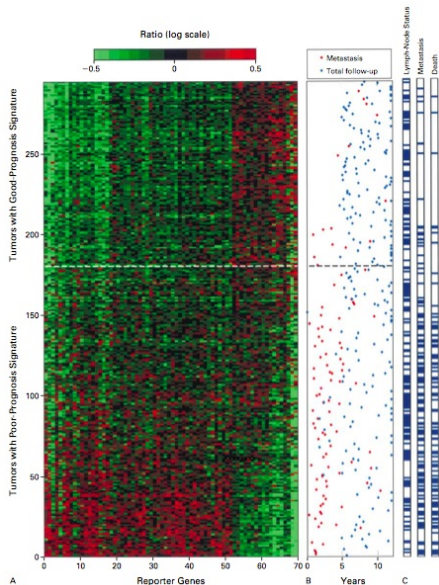


- CGH shows the (static) DNA
- Cancer cells have also **abnormal (dynamic) gene expression** (= transcription)

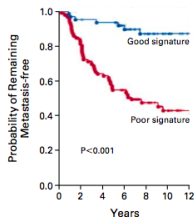
Can we identify the cancer subtype? (diagnosis)



Can we predict the future evolution? (prognosis)

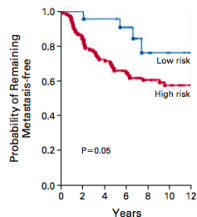


A Gene-Expression Profiling



No. AT RISK	0	2	4	6	8	10	12
Good signature	60	57	54	45	31	22	12
Poor signature	91	72	55	41	26	17	9

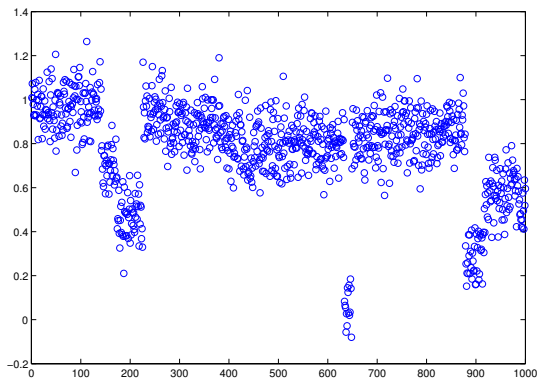
B St. Gallen Criteria



No. AT RISK	0	2	4	6	8	10	12
Low risk	22	22	21	17	9	5	2
High risk	129	107	88	69	48	34	19

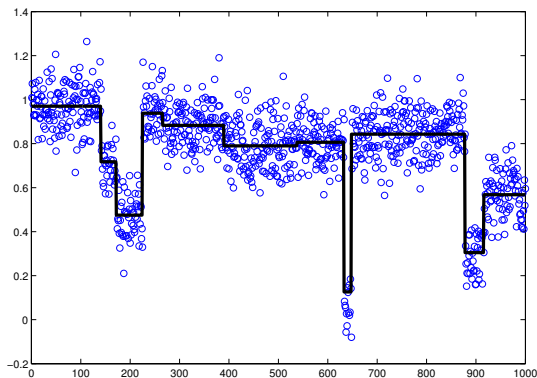
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The problem



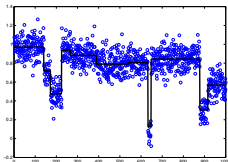
- Let $Y \in \mathbb{R}^p$ the signal
- We want to find a piecewise constant approximation $\hat{U} \in \mathbb{R}^p$ with at most k change-points.

The problem



- Let $Y \in \mathbb{R}^p$ the signal
- We want to find a piecewise constant approximation $\hat{U} \in \mathbb{R}^p$ with at most k change-points.

An optimal solution?

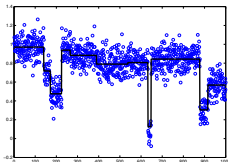


- We can define an "optimal" piecewise constant approximation $\hat{U} \in \mathbb{R}^p$ as the solution of

$$\min_{U \in \mathbb{R}^p} \|Y - U\|^2 \quad \text{such that} \quad \sum_{i=1}^{p-1} \mathbf{1}(U_{i+1} \neq U_i) \leq k$$

- This is an optimization problem over the $\binom{p}{k}$ partitions...
- Dynamic programming finds the solution in $O(p^2 k)$ in time and $O(p^2)$ in memory
- But: does not scale to $p = 10^6 \sim 10^9$...

An optimal solution?

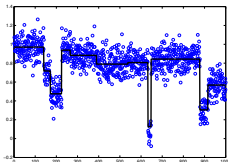


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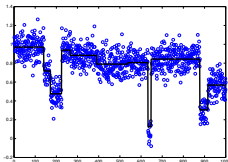


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Promoting sparsity with the ℓ_1 penalty

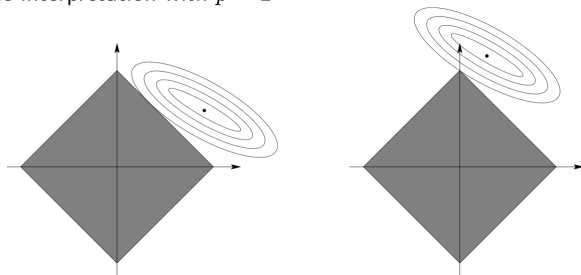
The ℓ_1 penalty (Tibshirani, 1996; Chen et al., 1998)

If $R(\beta)$ is convex and "smooth", the solution of

$$\min_{\beta \in \mathbb{R}^p} R(\beta) + \lambda \sum_{i=1}^p |\beta_i|$$

is usually **sparse**.

Geometric interpretation with $p = 2$



The total variation / variable fusion penalty

If $R(\beta)$ is convex and "smooth", the solution of

$$\min_{\beta \in \mathbb{R}^p} R(\beta) + \lambda \sum_{i=1}^{p-1} |\beta_{i+1} - \beta_i|$$

is usually piecewise constant (Rudin et al., 1992; Land and Friedman, 1996).

Proof:

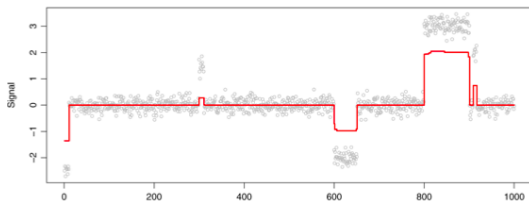
- Change of variable $u_i = \beta_{i+1} - \beta_i$, $u_0 = \beta_1$
- We obtain a Lasso problem in $u \in \mathbb{R}^{p-1}$
- u sparse means β piecewise constant

TV signal approximator

$$\min_{\beta \in \mathbb{R}^p} \|Y - \beta\|^2 \quad \text{such that} \quad \sum_{i=1}^{p-1} |\beta_{i+1} - \beta_i| \leq \mu$$

Adding additional constraints does not change the change-points:

- $\sum_{i=1}^p |\beta_i| \leq \nu$ (Tibshirani et al., 2005; Tibshirani and Wang, 2008)
- $\sum_{i=1}^p \beta_i^2 \leq \nu$ (Mairal et al. 2010)



Solving TV signal approximator

$$\min_{\beta \in \mathbb{R}^p} \|Y - \beta\|^2 \quad \text{such that} \quad \sum_{i=1}^{p-1} |\beta_{i+1} - \beta_i| \leq \mu$$

- QP with sparse linear constraints in $O(p^2)$ -> 135 min for $p = 10^5$ (Tibshirani and Wang, 2008)
- Coordinate descent-like method $O(p)$? -> 3s for $p = 10^5$ (Friedman et al., 2007)
- For all μ with the LARS in $O(pK)$ (Harchaoui and Levy-Leduc, 2008)
- For all μ in $O(p \ln p)$ (Hoefling, 2009)
- For the first K change-points in $O(p \ln K)$ (Bleakley and V., 2010)

TV signal approximator as dichotomic segmentation

Algorithm 1 Greedy dichotomic segmentation

Require: k number of intervals, $\gamma(I)$ gain function to split an interval I into $I_L(I), I_R(I)$

1: I_0 represents the interval $[1, n]$

2: $\mathcal{P} = \{I_0\}$

3: **for** $i = 1$ to k **do**

4: $I^* \leftarrow \arg \max_{I \in \mathcal{P}} \gamma(I)$

5: $\mathcal{P} \leftarrow \mathcal{P} \setminus \{I^*\}$

6: $\mathcal{P} \leftarrow \mathcal{P} \cup \{I_L(I^*), I_R(I^*)\}$

7: **end for**

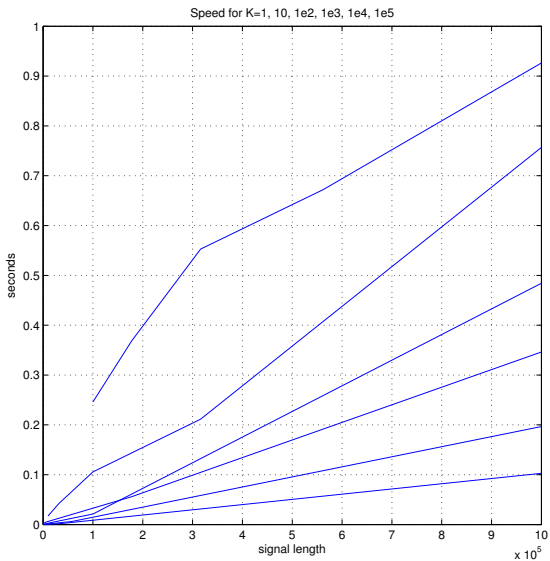
8: **return** \mathcal{P}

Theorem

TV signal approximator performs "greedy" dichotomic segmentation

(V. and Bleakley, 2010; see also Hoefling, 2009)

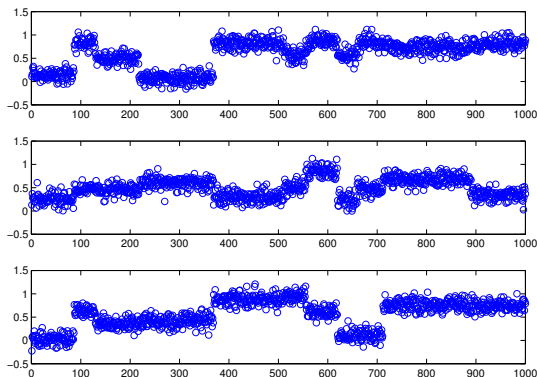
Speed trial : 2 s. for $K = 100$, $p = 10^7$



Outline

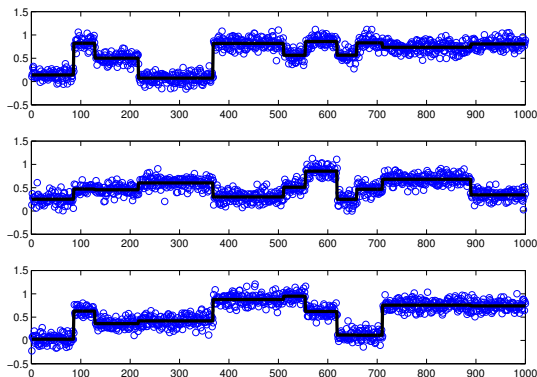
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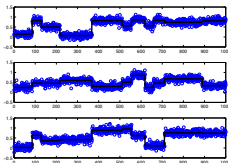
- Let $Y \in \mathbb{R}^{p \times n}$ the n signals of length p
- We want to find a piecewise constant approximation $\hat{U} \in \mathbb{R}^{p \times n}$ with at most k change-points.

The problem



- Let $Y \in \mathbb{R}^{p \times n}$ the n signals of length p
- We want to find a piecewise constant approximation $\hat{U} \in \mathbb{R}^{p \times n}$ with at most k change-points.

"Optimal" segmentation by dynamic programming



- Define the "optimal" piecewise constant approximation $\hat{U} \in \mathbb{R}^{p \times n}$ of Y as the solution of

$$\min_{U \in \mathbb{R}^{p \times n}} \|Y - U\|^2 \quad \text{such that} \quad \sum_{i=1}^{p-1} \mathbf{1}(U_{i+1, \bullet} \neq U_{i, \bullet}) \leq k$$

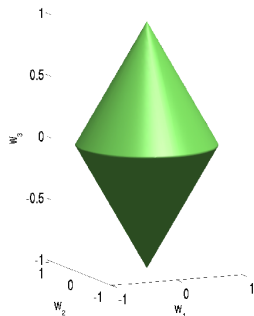
- DP finds the solution in $O(p^2 kn)$ in time and $O(p^2)$ in memory
- But: does not scale to $p = 10^6 \sim 10^9 \dots$

Selecting pre-defined groups of variables

Group lasso (Yuan & Lin, 2006)

If groups of covariates are likely to be selected together, the l_1/l_2 -norm induces sparse solutions *at the group level*:

$$\Omega_{group}(w) = \sum_g \|w_g\|_2$$



$$\begin{aligned}\Omega(w_1, w_2, w_3) &= \|(w_1, w_2)\|_2 + \|w_3\|_2 \\ &= \sqrt{w_1^2 + w_2^2} + \sqrt{w_3^2}\end{aligned}$$

TV approximator for many signals

- Replace

$$\min_{U \in \mathbb{R}^{p \times n}} \|Y - U\|^2 \quad \text{such that} \quad \sum_{i=1}^{p-1} \mathbf{1}(U_{i+1, \bullet} \neq U_{i, \bullet}) \leq k$$

by

$$\min_{U \in \mathbb{R}^{p \times n}} \|Y - U\|^2 \quad \text{such that} \quad \sum_{i=1}^{p-1} w_i \|U_{i+1, \bullet} - U_{i, \bullet}\| \leq \mu$$

Questions

- Practice: can we solve it efficiently?
- Theory: does it benefit from increasing p (for n fixed)?

TV approximator as a group Lasso problem

- Make the change of variables:

$$\begin{aligned}\gamma &= U_{1,\bullet}, \\ \beta_{i,\bullet} &= w_i (U_{i+1,\bullet} - U_{i,\bullet}) \quad \text{for } i = 1, \dots, p-1.\end{aligned}$$

- TV approximator is then equivalent to the following group Lasso problem (Yuan and Lin, 2006):

$$\min_{\beta \in \mathbb{R}^{(p-1) \times n}} \|\bar{Y} - \bar{X}\beta\|^2 + \lambda \sum_{i=1}^{p-1} \|\beta_{i,\bullet}\|,$$

where \bar{Y} is the centered signal matrix and \bar{X} is a particular $(p-1) \times (p-1)$ design matrix.

$$\min_{\beta \in \mathbb{R}^{(p-1) \times n}} \|\bar{Y} - \bar{X}\beta\|^2 + \lambda \sum_{i=1}^{p-1} \|\beta_{i,\bullet}\|,$$

Theorem

The TV approximator can be solved efficiently:

- **approximately** with the group LARS in $O(npk)$ in time and $O(np)$ in memory
- **exactly** with a block coordinate descent + active set method in $O(np)$ in memory

Proof: computational tricks...

Although \bar{X} is $(p-1) \times (p-1)$:

- For any $R \in \mathbb{R}^{p \times n}$, we can compute $C = \bar{X}^T R$ in $O(np)$ operations and memory
- For any two subset of indices $A = (a_1, \dots, a_{|A|})$ and $B = (b_1, \dots, b_{|B|})$ in $[1, p-1]$, we can compute $\bar{X}_{\bullet, A}^T \bar{X}_{\bullet, B}$ in $O(|A||B|)$ in time and memory
- For any $A = (a_1, \dots, a_{|A|})$, set of distinct indices with $1 \leq a_1 < \dots < a_{|A|} \leq p-1$, and for any $|A| \times n$ matrix R , we can compute $C = \left(\bar{X}_{\bullet, A}^T \bar{X}_{\bullet, A} \right)^{-1} R$ in $O(|A|n)$ in time and memory

Speed trial

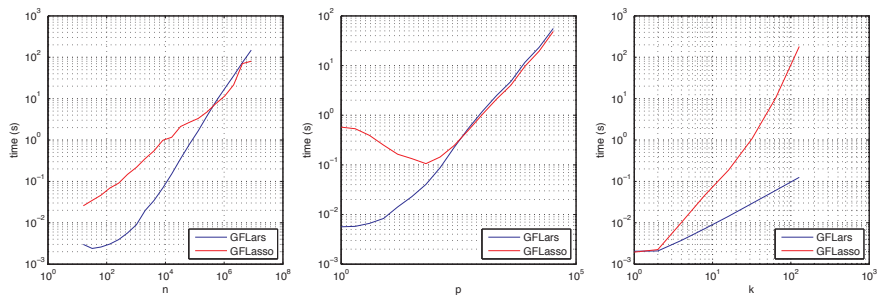
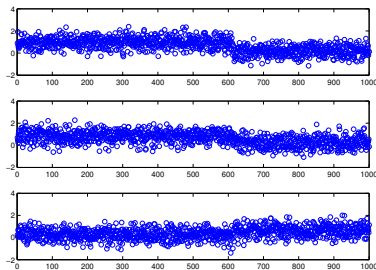


Figure 2: **Speed trials for group fused LARS (top row) and Lasso (bottom row).** *Left column:* varying n , with fixed $p = 10$ and $k = 10$; *center column:* varying p , with fixed $n = 1000$ and $k = 10$; *right column:* varying k , with fixed $n = 1000$ and $p = 10$. Figure axes are log-log. Results are averaged over 100 trials.

Consistency for a single change-point

Suppose a single change-point:

- at position $u = \alpha p$
- with increments $(\beta_i)_{i=1, \dots, n}$ s.t. $\bar{\beta}^2 = \lim_{k \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \beta_i^2$
- corrupted by i.i.d. Gaussian noise of variance σ^2



Does the TV approximator correctly estimate the first change-point as p increases?

Consistency of the unweighted TV approximator

$$\min_{U \in \mathbb{R}^{p \times n}} \|Y - U\|^2 \quad \text{such that} \quad \sum_{i=1}^{p-1} \|U_{i+1, \bullet} - U_{i, \bullet}\| \leq \mu$$

Theorem

The unweighted TV approximator finds the correct change-point with probability tending to 1 (resp. 0) as $n \rightarrow +\infty$ if $\sigma^2 < \tilde{\sigma}_\alpha^2$ (resp. $\sigma^2 > \tilde{\sigma}_\alpha^2$), where

$$\tilde{\sigma}_\alpha^2 = p\bar{\beta}^2 \frac{(1 - \alpha)^2 (\alpha - \frac{1}{2p})}{\alpha - \frac{1}{2} - \frac{1}{2p}}.$$

- correct estimation on $[p\epsilon, p(1 - \epsilon)]$ with $\epsilon = \sqrt{\frac{\sigma^2}{2p\bar{\beta}^2}} + o(p^{-1/2})$.
- wrong estimation near the boundaries

Consistency of the weighted TV approximator

$$\min_{U \in \mathbb{R}^{p \times n}} \|Y - U\|^2 \quad \text{such that} \quad \sum_{i=1}^{p-1} w_i \|U_{i+1, \bullet} - U_{i, \bullet}\| \leq \mu$$

Theorem

The weighted TV approximator with weights

$$\forall i \in [1, p-1], \quad w_i = \sqrt{\frac{i(p-i)}{p}}$$

correctly finds the first change-point with probability tending to 1 as $n \rightarrow +\infty$.

- we see the benefit of increasing n
- we see the benefit of adding weights to the TV penalty

- The first change-point \hat{i} found by TV approximator maximizes $F_i = \|\hat{c}_{i,\bullet}\|^2$, where

$$\hat{c} = \bar{X}^\top \bar{Y} = \bar{X}^\top \bar{X} \beta^* + \bar{X}^\top W.$$

- \hat{c} is Gaussian, and F_i follows a non-central χ^2 distribution with

$$G_i = \frac{EF_i}{p} = \frac{i(p-i)}{pw_i^2} \sigma^2 + \frac{\bar{\beta}^2}{w_i^2 w_u^2 p^2} \times \begin{cases} i^2 (p-u)^2 & \text{if } i \leq u, \\ u^2 (p-i)^2 & \text{otherwise.} \end{cases}$$

- We then just check when $G_u = \max_i G_i$

Consistency for a single change-point

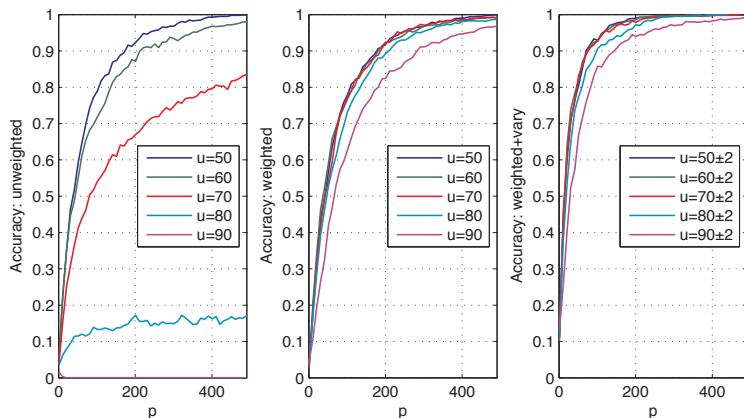


Figure 3: **Single change-point accuracy for the group fused Lasso.** Accuracy as a function of the number of profiles p when the change-point is placed in a variety of positions $u = 50$ to $u = 90$ (left and centre plots, resp. unweighted and weighted group fused Lasso), or: $u = 50 \pm 2$ to $u = 90 \pm 2$ (right plot, weighted with varying change-point location), for a signal of length 100.

Estimation of more change-points?

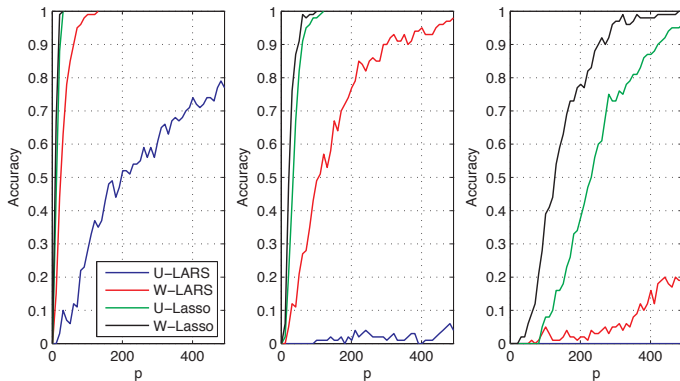
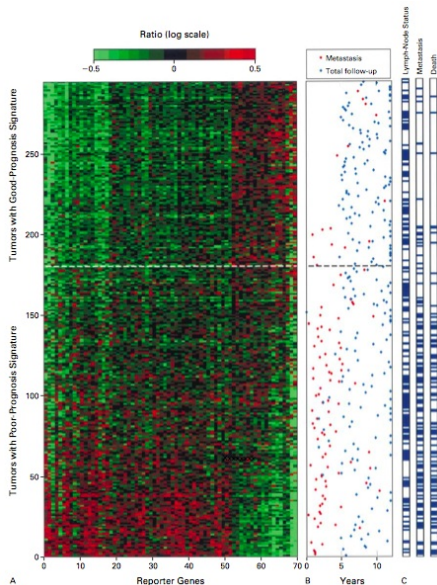
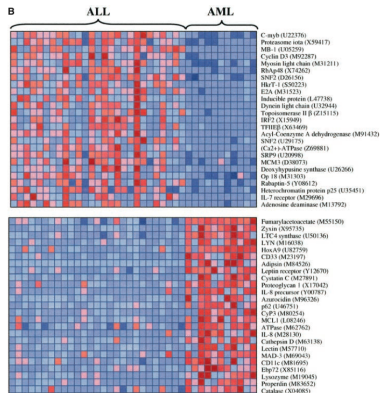


Figure 4: **Multiple change-point accuracy.** Accuracy as a function of the number of profiles p when change-points are placed at the nine positions $\{10, 20, \dots, 90\}$ and the variance σ^2 of the centered Gaussian noise is either 0.05 (left), 0.2 (center) and 1 (right). The profile length is 100.

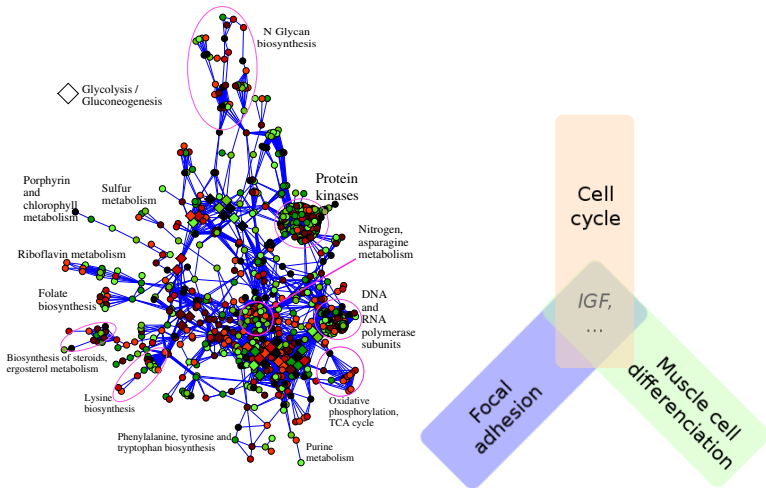
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Molecular diagnosis / prognosis / theragnosis

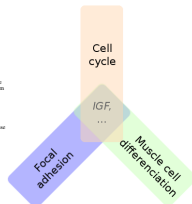
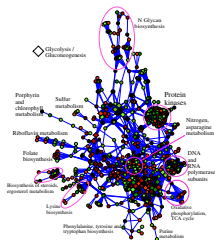
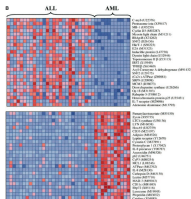


Gene networks, gene groups



Structured feature selection

- Basic biological functions usually involve the **coordinated action of several proteins**:
 - Formation of **protein complexes**
 - Activation of metabolic, signalling or regulatory **pathways**
- How to perform **structured feature selection**, such that selected genes
 - belong to only a few groups?
 - form a small number of connected components on the graph?

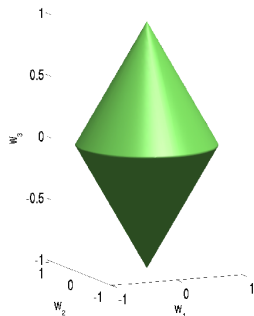


Selecting pre-defined groups of variables

Group lasso (Yuan & Lin, 2006)

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$$\Omega_{group}(w) = \sum_g \|w_g\|_2$$

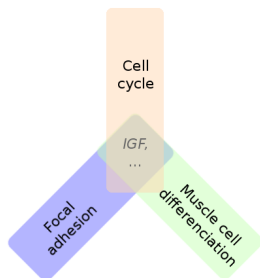


$$\Omega(w_1, w_2, w_3) = \|(w_1, w_2)\|_2 + \|w_3\|_2$$

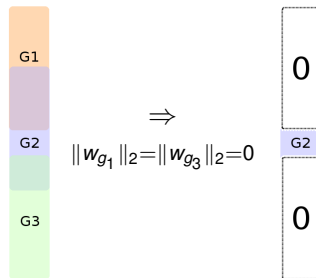
Group lasso with overlapping groups

Idea 1: shrink groups to zero (Jenatton et al., 2009)

- $\Omega_{group}(w) = \sum_g \|w_g\|_2$ sets groups to 0.
- One variable is selected \Leftrightarrow all the groups to which it belongs are selected.



IGF selection \Rightarrow selection of unwanted groups

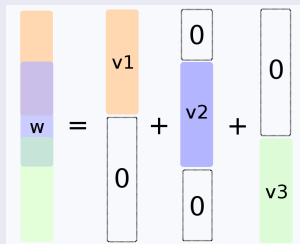


Removal of *any* group containing a gene \Rightarrow the weight of the gene is 0.

Group lasso with overlapping groups

Idea 2: latent group Lasso (Jacob et al., 2009)

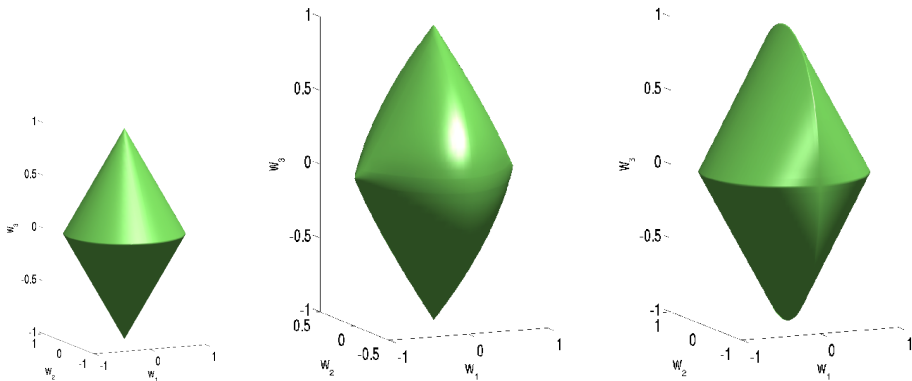
$$\Omega_{\text{latent}}^{\mathcal{G}}(w) \triangleq \begin{cases} \min_v \sum_{g \in \mathcal{G}} \|v_g\|_2 \\ w = \sum_{g \in \mathcal{G}} v_g \\ \text{supp}(v_g) \subseteq g. \end{cases}$$



Properties

- Resulting support is a *union* of groups in \mathcal{G} .
- Possible to select one variable without selecting all the groups containing it.
- Equivalent to group lasso when there is no overlap

Overlap and group unity balls



Balls for $\Omega_{\text{group}}^{\mathcal{G}}(\cdot)$ (middle) and $\Omega_{\text{latent}}^{\mathcal{G}}(\cdot)$ (right) for the groups $\mathcal{G} = \{\{1, 2\}, \{2, 3\}\}$ where w_2 is represented as the vertical coordinate. Left: group-lasso ($\mathcal{G} = \{\{1, 2\}, \{3\}\}$), for comparison.

Consistency in group support (Jacob et al., 2009)

- Let $\bar{\mathbf{w}}$ be the true parameter vector.
- Assume that there exists a unique decomposition $\bar{\mathbf{v}}_g$ such that $\bar{\mathbf{w}} = \sum_g \bar{\mathbf{v}}_g$ and $\Omega_{\text{latent}}^{\mathcal{G}}(\bar{\mathbf{w}}) = \sum \|\bar{\mathbf{v}}_g\|_2$.
- Consider the regularized empirical risk minimization problem $L(\mathbf{w}) + \lambda \Omega_{\text{latent}}^{\mathcal{G}}(\mathbf{w})$.

Then

- under appropriate mutual incoherence conditions on X ,
- as $n \rightarrow \infty$,
- with very high probability,

the optimal solution $\hat{\mathbf{w}}$ admits a unique decomposition $(\hat{\mathbf{v}}_g)_{g \in \mathcal{G}}$ such that

$$\{g \in \mathcal{G} | \hat{\mathbf{v}}_g \neq \mathbf{0}\} = \{g \in \mathcal{G} | \bar{\mathbf{v}}_g \neq \mathbf{0}\}.$$

Consistency in group support (Jacob et al., 2009)

- Let \bar{w} be the true parameter vector.
- Assume that there exists a unique decomposition \bar{v}_g such that $\bar{w} = \sum_g \bar{v}_g$ and $\Omega_{\text{latent}}^{\mathcal{G}}(\bar{w}) = \sum \|\bar{v}_g\|_2$.
- Consider the regularized empirical risk minimization problem $L(w) + \lambda \Omega_{\text{latent}}^{\mathcal{G}}(w)$.

Then

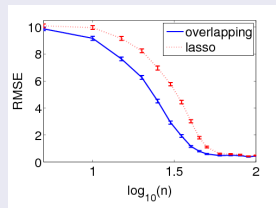
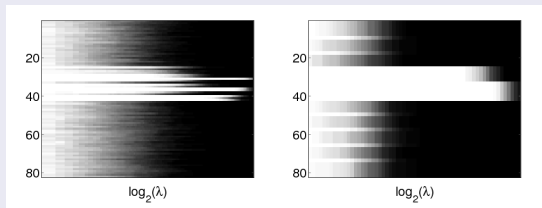
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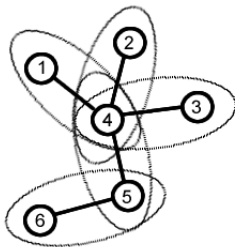
$$\{g \in \mathcal{G} | \hat{v}_g \neq 0\} = \{g \in \mathcal{G} | \bar{v}_g \neq 0\}.$$

Synthetic data: overlapping groups

- 10 groups of 10 variables with 2 variables of overlap between two successive groups : $\{1, \dots, 10\}, \{9, \dots, 18\}, \dots, \{73, \dots, 82\}$.
- Support: union of 4th and 5th groups.
- Learn from 100 training points.



Frequency of selection of each variable with the lasso (left) and $\Omega_{\text{latent}}^{\mathcal{G}}(\cdot)$ (middle), comparison of the RMSE of both methods (right).



Two solutions

$$\Omega_{\text{group}}^{\mathcal{G}}(\beta) = \sum_{i \sim j} \sqrt{\beta_i^2 + \beta_j^2},$$

$$\Omega_{\text{latent}}^{\mathcal{G}}(\beta) = \sup_{\alpha \in \mathbb{R}^p: \forall i \sim j, \|\alpha_i^2 + \alpha_j^2\| \leq 1} \alpha^{\top} \beta.$$

Breast cancer data

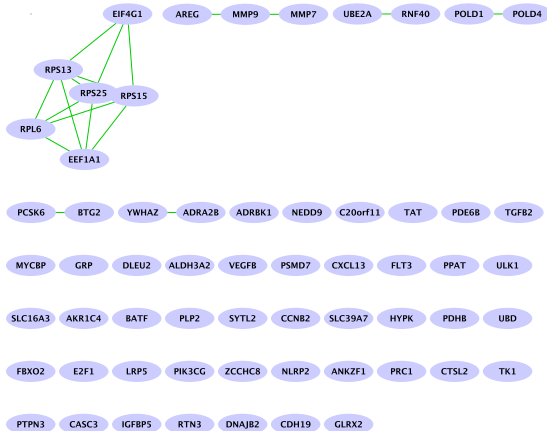
- Gene expression data for 8,141 genes in 295 breast cancer tumors.
- Canonical pathways from MSigDB containing 639 groups of genes, 637 of which involve genes from our study.

METHOD	l_1	$\Omega_{\text{LATENT}}^G(\cdot)$
ERROR	0.38 ± 0.04	0.36 ± 0.03
MEAN $\#$ PATH.	130	30

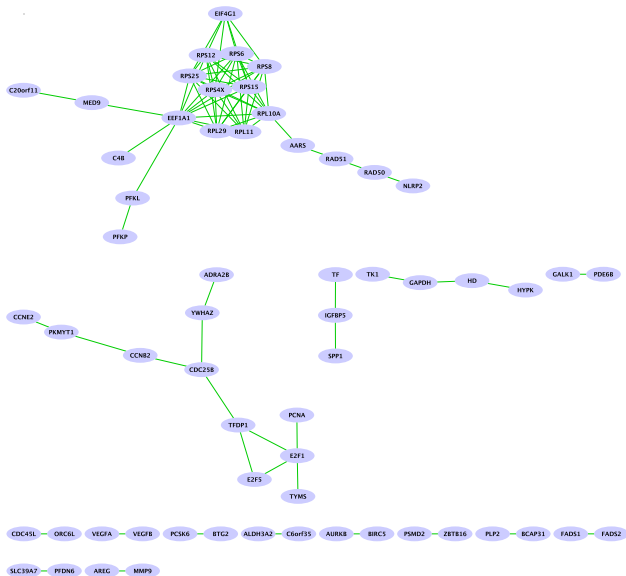
- Graph on the genes.

METHOD	l_1	$\Omega_{\text{graph}}(\cdot)$
ERROR	0.39 ± 0.04	0.36 ± 0.01
AV. SIZE C.C.	1.03	1.30

Lasso signature



Graph Lasso signature



Outline

- 1 Motivations
- 2 Finding multiple change-points in a single profile
- 3 Finding multiple change-points shared by many signals
- 4 Learning molecular classifiers with network information
- 5 Conclusion**

Conclusions

- Penalty design as a way to incorporate prior knowledge
- Convex sparsity-inducing penalties are useful; efficient implementations + consistency results



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Post-docs available in Paris!



European Research Council

